Benacerraf's Mathematical Antinomy and Its Kantian Solution

Workshop *Reconciling Nominalism and Platonism*
November 27, 2015

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Reconciling: What about and what for?

Before reconciling Nominalism and Platonism in philosophy of mathematics, it is quite in order to identify their disagreement.

It is quite fair as well to consider Benacerraf's dilemma as a very clear and influential formulation of this disagreement.

In Benacerraf's rephrasing, the Nominalist is actually called a "formalist," and Benacerraf claims to present a somewhat generalized version of the disagreement, where the Platonist and the Nominalist epitomize more general views about mathematical truth.
All the more, Benacerraf's dilemma is a natural starting point.

Given the disagreement expressed by Benacerraf's dilemma, why should one try to overcome it?
I think the answer simply is: because this dilemma (and others of that ilk) should have been overcome more quickly.

Paul Benacerraf, "Mathematical Truth" (1973)

The main issue is about the right construal of mathematical statements:

- Either they are taken at face value, i.e. as being directly *about* certain objects, according to a *referentialist* semantical account. This is what Benacerraf calls the "semantical" account (= Tarski-Gödel).
- Or they are considered as embedded in a symbolic calculus, and their assertion boils down to their formal derivability from certain axioms. This is what Benacerraf calls the "combinatorial" account (= Hilbert) – insofar as truth values are assigned on the basis of purely syntactic (proof-theoretic) facts.

Benacerraf's example:

(1) There are at least three large cities older than New York.
(2) There are at least three perfect numbers greater than 17.

If one construes (2) as sharing the same kind of truth conditions as (1), then one is committed to a referential parsing that aligns numbers on regular objects. Mathematical truth is just one case of ordinary truth.
But, then, how to explain our epistemic access to such entities that numbers are supposed to be?

On the contrary, if one completely rephrases (2) in formal terms, using the axiomatic system of Peano arithmetic or the classical (second-order) Frege-Russell analysis, then one ceases to put (2) on a par with (1), and the epistemic access is no more problematic, since the manipulation of symbols according to rules is obviously available to us.
But, then, how to explain that a mathematical sentence is a truth, beyond being merely a formal theorem?

Two conditions are the standards for any "over-all view" of mathematical truth.

The first condition is

> *[...] the requirement that there be an over-all theory of truth in terms of what it can be certified that the account of mathematical truth is indeed an account of mathematical truth. The account should imply truth conditions for mathematical propositions that are evidently conditions conditions of their truth (and not simply, say, of their theoremhood on some formal system). (p. 666)*

What Benacerraf has in mind is a uniform theory of truth for the language as a whole, including the mathematical language as a mere sub-language.

In Benacerraf's reckoning, only a Tarskian referential semantics can do the job.

Second condition for any "over-all view" of mathematical truth:

> *My second condition on an over-all view presupposes that we have mathematical knowledge. [...] The minimal requirement, then, is that a satisfactory account of mathematical truth must be consistent with the possibility that some such truths be knowable. [...] An acceptable semantics for mathematics must fit an acceptable epistemology. (p. 667)*

According to Benacerraf, the "combinatorial account" precisely stems from epistemological concerns.

Reducing mathematical knowledge to a formal proof activity avoids having to deal with non-empirical objects.

Summary of Benacerraf's dilemma:

Benacerraf's point is that you cannot fulfill both conditions at the same time.

- Either you endorse the semantical account, and then you have a uniform semantics for ordinary language extended to mathematical language, but then one lapses into platonism.
- Or you endorse the combinatorial account, and then you have a reasonable epistemology of mathematical knowledge as a proof activity, but no account of mathematical truth other than formal is given.

So you cannot have it both ways: You have to choose between truth and knowledge.

However, truth without knowledge or knowledge without truth are both self-defeating options.
Benacerraf's text does not provide any clear solution. Hence a dilemma.

Benacerraf about the combinatorial account:

> *[...] there is little mystery about how we can obtain mathematical knowledge. We need only account for our ability to product and survey formal proofs. However, squeezing the balloon at that point apparently makes it bulge on the side of truth: the more nicely we tie up the concept of proof, the more closely we link the definition of proof to combinatorial (rather than semantical) features, the more difficult it is to connect it up with the truth of what is being thus "proved" – or so it would appear. (p. 668)*

Miscellaneous remarks:

- The dichotomy is not one (e.g., what about Russell, for instance?).
- The semantical account is very weak (but Benacerraf's paper precisely tilts the scales so as to get an embarrassing balance, and thus a dilemma).
- The "semantical" account is not the only semantical account possible: Benacerraf adds subreptitiously the requirement that the semantics has to be compositional and uniform.
- Explaining the referentiality of a mathematical theory, namely granting that that theory is not simply wheelspinning, does not require to abide by the superficial grammar. (And Benacerraf knows that very well.)
- The epistemology is very naive (very basic empirical logicism).

- The semantical account relies on set theory. But then the actual semantical value of the set-theoretic terms that any Tarskian-style semantics resorts to, should be accounted for. Hence an infinite regress (the problem is only pushed back one square higher up).
- Mathematical theories are neither descriptive nor purely formal. Mathematics evades this duality. Mathematical objectivity is neither "over there" nor fictitious.
- "Benacerraf's dilemma" has been, ENORMOUSLY, the focus of analytic philosophy of mathematics since it has been written.
- It also has put off French philosophers of mathematics as simplistic and out of touch with real mathematics.

Idea: Benacerraf's dilemma evokes Kant's Antinomies of Pure Reason, and more specifically the "mathematical" ones.

Reminder: Both opponents of Kant's two first antinomies (i.e., of the "mathematical" ones) are wrong, whereas both opponents of the two last antinomies (i.e., of the " dynamical" ones) are right (albeit from two different points of view).

Claim: A comparison between Benacerraf's dilemma and Kant's mathematical antinomies is called for by strong analogies, and should be useful, since Kant, in addition to presenting a predicament analogous to Benacerraf's dilemma, does provide a solution for it.

Goal: Harnessing the analogy with Kant so as to transpose Kant's solution to Benacerraf's setting (and thus solve Benacerraf's dilemma).

Mathematical Antinomies of Pure Reason

1. First Antinomy (A426-427/B454-455)
   - Thesis: "The world has a beginning in time, and in space it is also enclosed in boundaries."
   - Antithesis: "The world has no beginning, and no bounds in space, but is infinite with regard to both time and space."
2. Second Antinomy (A434-435/B462-463):
   - Thesis: "Every composite substance in the world consists of simple parts, and nothing exists anywhere except the simple or what is composed of simples."
   - Antithesis: "No composite thing in the world consists of simple parts, and nowhere in it does there exist anything simple."

Comments on the Kantian antinomies

In all Kantian antinomies, each claim is mainly negative: it relies on a reductio ad absurdum and feeds entirely upon the impossibility of the opposite claim.

In the same way, each horn of Benacerraf's dilemma draws its strength only from the predicament of the other.

Each Kantian antinomy is about the concept of something unconditioned with respect to some condition.

There are actually two different ways of conceiving of the unconditioned (A417/B445): either as being the last term of the regressive series of conditions, or as consisting in the whole series itself.

In each antinomy, the thesis epitomizes the first conception, the antithesis the second one.

The thesis seeks a first unconditioned *entity* on which the whole series of conditions depends: It is the reason trying to catch up with the understanding, since the unconditioned is presented as an actual object.

In a reversal from that, the antithesis presents the sum of the conditions in the series as constituting an unconditioned totality: it is the understanding trying to catch up with reason.

As a consequence, cosmological ideas are either too large or too small for the empirical regress sustained by the concepts of the understanding (mismatch between understanding and reason).

[Assume] that the world has no beginning: then it is too big for your concept; for this concept, which consists in a successive regress, can never reach the whole eternity that has elapsed.
Suppose it has a beginning, then once again it is too small for your concept of understanding in the necessary empirical regress. For since the beginning always presupposes a preceding time, it is still not unconditioned, and the law of the empirical use of the understanding obliges you to ask for a still higher temporal condition, and the world is obviously too small for this law.
(A486-487/B514-515)

In each antinomy:

- The thesis aims at some absolute entity (what Kant calls an *Object*, as opposed to a *Gegenstand*).
  Thesis = dogmatism
- The antithesis sticks to the limits of sensible experience: The relationships between appearances and the laws of those relationships are the main focus.
  Antithesis = empiricism

THE ANALOGY (not a resemblance) is:

- between
  1. the stress put by empiricism on the understanding (in Kant)
  2. and the stress put by the combinatorial account on mathematical proofs (in Benacerraf)
- between
  1. the stress put by dogmatism on *Objects* (in Kant)
  2. and the stress put by the semantical account on mathematical objects (in Benacerraf).

In other words:

- Kantian understanding (characterized as the faculty of rules) $\rightsquigarrow$ production of formal proofs by mathematics
- Kantian reason (characterized as the quest for unconditioned entities) $\rightsquigarrow$ reference of mathematics to an actual non-empirical denotation.

To get back to the first antinomy:

- Instead of saying that space has a definite extension, Benacerraf's dogmatic (= the semanticist) claims that mathematical terms actually refer to a definite entity.
- Instead of saying that space is boundless, Benacerraf's empiricist (= the combinatorialist) claims that a mathematical object is nothing but the open-ended sum of all the formal proofs that we can produce about it.

Analogy between the Antinomy of Pure Reason and Benacerraf 1973:

Kant	Benacerraf
mathematical antinomy	dilemma
antithetic (each thesis feeds upon the contradiction of the other)	negative argumentation
understanding	"our ability to produce and survey formal proofs" (p. 409)
appearances	symbols
possible experience	admissible inference
series of conditions	deductive chains
law established by the understanding	theorem

Analogy cont'd:

Kant	Benacerraf
reason	truth theory
unconditioned being	direct reference
intellectual intuition	Gödelian intuition
dogmatism (thesis)	realism (semantical conception)
Platonism	platonism (in mathematics)
empiricism (antithesis)	finitism (combinatorial conception)
Epicureanism	Hilbertian formalism

Analogy cont'd:

Kant	Benacerraf
The Idea is either too big or too small for the concept of the understanding (A486/B514).	All analyses "bulge either on the side of knowledge or on the the side of truth" (p. 668).
The antithesis favors knowledge of nature, at the cost of the practical (A469-472/B497-500).	The combinatorial account explains mathematical knowledge, at the cost of mathematical referentiality.
The thesis meets the practical interest of reason, but neglects the investigation of nature.	The semantical account dovetails with the pragmatic need of a unified referential framework, but lets the objects prevail over one's possible access to them.
Solution provided by transcendental idealism	*To be specified*

The detour through Kant's Transcendantal Dialectic calls forth two important points:

- FIRST POINT: "Benacerraf's antinomy" is of the mathematical kind, not of the dynamical one.
- SECOND POINT: Kant's treatment hints at a solution of Benacerraf's dilemma itself, since Kant provided a systematic and extensive solution for the antinomies that can be transposed.

FIRST POINT: Benacerraf's dilemma is analogous to a *mathematical* antinomy

It could seem, indeed, that Benacerraf's dilemma is a dynamical antinomy, as though mathematical truth could be looked at from the point of view of understanding (epistemology) as well as from that of reason (semantics), so that both claims would be legitimate within their respective limits.

Against that view, Benacerraf's dilemma must clearly be understood as an antinomy of the mathematical kind, where BOTH opposite claims are WRONG.

Actually, Benacerraf makes it plain that neither account is philosophically and that both accounts must be overcome.

- Benacerraf, "What numbers could not be" (1965): The semantical account is proved to be wrong.
- Benacerraf, "Frege: The Last Logicist" (1981): The combinatorial account is proved to be wrong.

Benacerraf 1965: about two competing accounts of natural numbers (Cantor's and von Neumann's)

This paper clearly shows that the semantical values of numerical terms like '3' or '17' are not univocal. Even when we are using genuine singular terms in mathematics, their reference is not set unambiguously.

The basic lesson to be learned is that neither account can provide the right semantical value for natural numbers.

In fact, both accounts can be construed as being, precisely, two different interpretations of the same mathematical objects.

Cantor's version and von Neumann's correspond to two different structures C and N for $\mathrm{L} = \{\in, \underline{0}, S\}$:

- C is the L-structure whose domain is $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \ldots\}$
- N is the L-structure whose domain is $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \ldots\}$.

It is a standard set-theoretic result that C and N are *mutually interpretable* (in the model-theoretic sense): Cantor's ordinals are the transitive closures of von Neumann's.

Cantor's version and von Neumann's versions = two different presentations of the same mathematical objects.

A mathematical objects like 3 corresponds in fact to two different items in two different models (e.g., $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ in the model C, as well as $\{\{\{\emptyset\}\}\}$ in the model N), supplemented with the proof that the two models are mutually interpretable and that the interpretations at stake relate the two items to each other.

This is true of "the" natural numbers, which have to be understood as the invariant of a series of equivalent models ("equivalent" in some sense to be specified) – the proof of that equivalence being integral to "the natural numbers" themselves.

Quite generally, a mathematical object emerges as the invariant of the series of all its possible presentations, and intrinsically involves the *proof* that all those presentations are presentations of the same thing. This proof is built into the mathematical object as such.

Conclusion: Objects involve proofs. So the semantical account is inconsistent.

Benacerraf 1981: claims that the whole enterprise of Frege's *Grundlagen* is "first and foremost a mathematical one."

Grundlagen, §3:

> *[. . .] the question [as to whether a proposition is a priori or not] is removed from the sphere of psychology, and assigned, if the truth concerned is a mathematical one, to the sphere of mathematics.*

Benacerraf's comment (p. 55):

> *Since arithmetical propositions are at issue, the question of their justification is properly a matter for mathematics. Therefore, the concepts will be so defined as to make it a properly mathematical question whether some arithmetical judgment is analytic or synthetic, a priori or a posteriori.*

The question as to whether a given mathematical proposition is analytic becomes a mathematical one, not only because "the truth concerned is a mathematical one," but because the way of establishing its analyticity is mathematical.

The recognition of special mathematical truths as being analytical requires to turn proofs themselves into mathematical objects in their own right, which are no less epistemically problematic than the number 3.

Conclusion: Proofs can (have to) be turned into objects. So the combinatorial account is inconsistent as well.

So finally BOTH accounts are WRONG.

CONCLUSION OF THE FIRST POINT:

Benacerraf's dilemma = *mathematical* antinomy (in the Kantian sense) about mathematics

SECOND POINT: Transposition of Kant's solution to Benacerraf's dilemma

One of the main benefits that can be expected from the analogy made between Kant's *Dialectic* and Benacerraf's dilemma is to provide a clue for a solution of the latter.

Benacerraf's 1965 and 1981 papers already suggest the clue: in Benacerraf's dilemma, both opposite claims are false for the same reason (*mutatis mutandis*) as in Kant:
Both opponents take mathematical items as given in themselves.

- "Space has a bounded extension" (dogmatism) $\rightsquigarrow$ Mathematical objects are what they are, once for all.
- "Space is boundless" (empiricism) $\rightsquigarrow$ A mathematical object is nothing but the illimited sum of all the proofs that we can produce about it and does not exist beyond those proofs.

Kant's solution of the first antinomy:

> *If one regards the two propositions 'The world is infinite in magnitude,' 'The world is finite in magnitude,' as contradictory opposites, then one assumes that the world (the whole series of appearances) is a thing in itself. [. . .] But if I take away this presupposition, or rather this transcendental illusion, and deny that it is a thing in itself, then the contradictory conflict of the two assertions is transformed into a merely dialectical conflict, and because the world does not exist at all (independently of the regressive series of my representations), it exists neither as an in itself infinite whole nor as an in itself finite whole. It is only in the empirical regress of the series of appearances, and by itself it is not to be met with at all. (A504/B532)*

The whole antinomy of pure reason relies on the false assumption that the objects of experience are given *in themselves*, whereas they are given only in the course of a regressive series of conditions.

Kant's solution transposed: Mathematical objects are not things in themselves.

They are neither full-fledged existing entities nor mere fictions introduced in the course of a proof.

That mathematical objects are not given in themselves means something quite basic, namely that any mathematical object goes with modes of presentation whose nature depends on the kind of object at stake.

Caveat

The notion of presentation that I introduced 3 weeks ago mixed up two different things: the notion of setting that I introduced then (as exemplified in the context of combinatorics and graph theory) and the notion of mode of presentation that I am introducing now.

This notion of mode of presentation is different from my former notion of presentation and supersedes everything I may have said 3 weeks ago.

Modes of presentation

Various examples of multiple modes of presentation abund in mathematics, on different scales:

- Natural numbers can be defined as (Ernie) Zermelo does, or as (Johnny) von Neumann does.
- A vector space is usually presented through an affine space fixed by the arbitrary choice of some origin.
- The set of complex numbers can be defined algebraically as $\mathbb{R}/(X^2+1)$, arithmetically as the set $\{a+ib : a, b \in \mathbb{R}\}$ endowed with addition and multiplication, geometrically as points of the plane.
- Examples in algebra: "presentation" of a group by generators and relations (I will get back to this example in a moment), resolution of a module.

Admittedly, different scales of presentation ought to be distinguished, because in some cases the "same" structure is introduced with the help of different supports (as in the case of complex numbers), whereas in some other cases differents structures provide different accounts of the "same" mathematical concept (as in the case of the Ernie *vs* Johnny controversy)

Yet these differences do not detract from the occurrence of a same phenomenon, namely the diffraction of a "same" mathematical item into various modes of presentation, without which this mathematical item cannot be grasped, let alone studied.

Such is already, in a way, Benacerraf's diagnosis, as early as 1965, before the dilemma was properly coined:

> *Any purpose we may have in giving an account of the notion of number and of the individual numbers, other than the question-begging one of proving of the right set of sets that it is the set of numbers, will be equally well (or badly) served by any one of the infinitely many accounts satisfying the conditions we set out so tediously. Benacerraf 1965, p. 284)*

A mathematical object can never be given “in itself.”

Most mathematical structures never go without some symbolic devices which allows one to set ideas and without whose bias the intended structure cannot be reached, i.e., cognitively handled.

Bringing up the notion of mathematical “mode of presentation” (or “account,” to use Benacerraf’s term) is a general way to single out the pervasive use of such devices throughout mathematics.

Modes of presentation do not boil down to sub-mathematical conditions of concrete mathematical activity, such as the actual drawing compared to the geometrical theorem. They are truly mathematical in nature and lend themselves sometimes to an explicit mathematical treatment, as shown by the notion of presentation of a group by generators and relations.

About the very phrase “mode of presentation”

This term was mentioned by Frege in the context of the distinction that he drew between sense and denotation:

> *If the sign ‘a’ is distinguished from the sign ‘b’ only as object (here, by means of its shape), not as sign (i.e. not by the manner in which it designates something), the cognitive value of $a = a$ becomes essentially equal to that of $a = b$, provided $a = b$ is true. A difference can arise only if the difference between the signs corresponds to a difference in the mode of presentation of that which is designated. Let a, b, c be the lines connecting the vertices of a triangle with the midpoints of the opposite sides. The point of intersection of a and b is then the same as the point of intersection of b and c. So we have different designations for the same point, and these names (‘point of intersection of a and b’, ‘point of intersection of b and c’) likewise indicate the mode of presentation [Art des Gegebenseins]; and hence the statement contains actual knowledge.*

The choice that Frege made of the term "presentation" could certainly be driven back to two opposite sources:

- the work of Franz Brentano in psychology, in particular his book *Psychology from an Empirical Standpoint* (1874), where the notion of "mode of presentation" (*Modus des Vorstellens*) is ubiquitous
- the foundation of the theory of group presentations by Walther von Dyck (a student of Felix Klein) in his article "Gruppentheoretische Studien," published in 1882 in the *Mathematische Annalen*.

The Fregean notion of mode of presentation should be construed as both epistemological and logical, both as an epistemic access, and as a presentation in the mathematical sense of the presentation of a group.

The mathematical example taken by Frege in the quote must be understood as a way to introduce a notion that is a common platform to account for both the basic phenomenon of meaning in ordinary language and the customary use of various descriptions of mathematical objects.

About the notion of mode of presentation

The notion of *mode of presentation* is a tentative notion.
For sure, mathematical modes of presentation take on various aspects and their spectrum is hardly amenable to a single kind.

In particular, modes of presentation are certainly many-layered: The mode of presentation of some object can itself be turned into an object w.r.t. some of its own modes of presentation.

Let us focus on one example: the notion of "presentation of a group by generators and relations".

Let $a, b, c, \dots$ be given symbols. A word in these symbols is any finite sequence

$$f_1 f_2 \dots f_{n-1} f_n$$

where each f_i is one of the symbols $a, b, c, \dots, a^{-1}, b^{-1}, c^{-1}, \dots$

By convention, the empty word is assumed to exist and is denoted by 1.

The inverse of the word $f_1 f_2 \dots f_{n-1} f_n$ is defined to be $f_n^{-1} f_{n-1}^{-1} \dots f_2^{-1} f_1^{-1}$.

Concatenation endows the set of all words with a structure of group: This is the "free group" generated by $a, b, c, \dots$

A relation is any equality $W = 1$, where $W(a, b, c, \dots)$ is a word.

The *trivial relations* are all relations $aa^{-1} = 1, a^{-1}a = 1, bb^{-1} = 1, b^{-1}b = 1, \dots$

Given a set $P = 1, Q = 1, R = 1, \ldots$ of relations (including the trivial ones), two words are equivalent if each one can be obtained from the other by either inserting or deleting one of the words $P, P^{-1}, Q, Q^{-1}, R, R^{-1}, \ldots$

The quotient of the free group of words by this equivalence relation is called the group presented by the generators $a, b, c, \ldots$ and the relations $P = 1, Q = 1, R = 1, \ldots$ The presentation is written:

$$[a, b, c, \ldots | P, Q, R, \ldots] \; .$$

Examples:

- $[a, a^n = 1]$ presents the cyclic group $\mathbb{Z}/n\mathbb{Z}$.
- $[r, f | r^4, f^2, fr = r^{-1}f]$ presents the dihedral group D_4.
- $[i, j | jij = i, iji = j]$ presents the group of quaternions.
- Every group has a presentation.

Very importantly, $[a, a^n = 1]$ IS (a presentation of) the n-th cyclic group $\mathbb{Z}/n\mathbb{Z}$.

It is neither a mere sequence of symbols (as the formalist would have it), because it actually presents something; nor a name (as the platonist would have it), because it has an internal structure that is by itself informative and does not point to any external object.

Tracing back "the" n-th cyclic group to the presentation $[a, a^n = 1]$ shows how one can deal (and usually deals) with the former on the much more cognitively tractable basis of the latter.

Of course, other presentations of the n-th cyclic group are available, in particular modes of presentation that are not specifically presentations by generators and relations.
The task of mathematics is precisely to link all the known modes of presentation together as equivalent presentations of the same "thing."

Kant's solution transposed

Just as the world really is something, but only as an incomplete series of conditions and as the regulative focus thereof on "the world,"
in the same way a mathematical object actually corresponds to an open-ended bundle of provably equivalent modes of presentation (i.e., an open-ended bundle of equivalence proofs about modes of presentation).

The mathematical object "itself" is but the regulative invariant of the *open-ended* series of all its possible modes of presentation, which themselves are neither purely formal items nor independent semantical units.

In a way, it is only about looking at the symbolic nature of mathematics without clinging to a proof-theoretic straight jacket.

Kant's solution transposed, cont'd

Just as the cosmological idea, in Kant, "is only in the empirical regress of the series of appearances, and by itself it is not to be met with at all" (A505/B533),
in the same way a mathematical structure never exists beyond the series of its presentations, none of which can be privileged as giving what would seem to be "the structure itself."

And just as the unreachable completion of a series of conditions is the task prescribed to the understanding as a "regulative principle of reason,"
in the same way the never-ending exploration of all possible presentations of a "same" mathematical structure (definitions becoming theorems, and vice versa), which amounts to the establishment of all the possible theorems about it, is the *task*, never amenable to completion, that defines mathematics as a discipline.

The presentational view that I am defending claims that mathematical objectivity is intentional, insofar as a coherent bundle of presentations points to an object, but only in a regulative way, without positing any pointee.

On the contrary, the semantical account considers that any pointing at all presupposes a pointee that exists by itself,

whereas the combinatorial account denies any pointing at all and considers a series of presentations, in and of itself, to be all that there is.

Both accounts miss the crucial fact that doing mathematics is constantly shifting from one presentation to another, provably equivalent, one.
This fact is not only witnessed by history of mathematics and by mathematical practice, but constitutes the very core and texture of mathematics.

- The mistake of the combinatorial account is the wrong thesis that mathematical presentations do not present anything.
- The mistake of the semantical account —the mistake of contemporary structuralism (as shown by the "identity problem" raised by Jukka Keränen against Stewart Shapiro)— is the wrong thesis that presentations are mere artefacts as opposed to the mathematical structures "themselves."

Taking modes of presentation seriously, as essential devices in-between mathematical structures and the empirical signs that symbolize them is a way to overcome Benacerraf's dilemma.

In the same way, taking modes of presentation seriously, as essential devices in-between mathematical structures and the empirical systems that instanciate them, would be a way to overcome the identity problem.

One last point

As mentioned above, Kant explains that, in the first antinomy, the idea of the world is too small for the concept of the understanding in the thesis, and too big for it in the antithesis.

In the case of Benacerraf's dilemma, one could be tempted to say the other way around: that in the thesis (the semantical account) the idea of mathematical objectivity is too big for the understanding, and that in the antithesis (the combinatorial account) it becomes too small.

Benacerraf himself writes:

> *[...] a typical 'standard' account (at least in the case of number theory or set theory) will depict truth conditions in terms of conditions on objects whose nature, as normally conceived, places them beyond the reach of the better understood means of human cognition (e.g., sense perception and the like).*
> *[...] postulational stipulation makes no connection between the propositions and their subject matter – stipulation does not provide truth. At best, it limits the class of truth definitions (interpretations) consistent with the stipulations. But that is not enough. (Benacerraf 1973, p. 667-668 and p. 679)*

The semantical account should be said too wide and yet, in Kant's transposition, it is said to be too narrow.
The same about the combinatorial account.

How to explain this apparent inversion?

In fact, there is none.

The idea of mathematical objectivity, as supplied by the semantical account, is really too small, and the same idea, as supplied by the combinatorial account, really too big.

The yardstick, indeed, is not so much our cognitive power as an open-ended series of presentations that mathematicians come up with in history.

The semantical account closes the process too early (and thus supports too narrow an idea of a mathematical object): The content of a mathematical concept is fixed once for all – although it continues being enriched by new theorems, so that the problem becomes to decide whether one keeps the same object when some really new presentation of it is put forward.

On the contrary, the combinatorial account contends that a mathematical object is already nothing else than the (thereby too big) complete series of all that is and will be proved about it, which now raises the symmetric problem of accounting for genuine ruptures in history of mathematics.

Both opponents neglect the deep historical nature of mathematical objectivity.

This should come as no surprise, since the main representants of both sides (Tarski or Gödel, and Hilbert, respectively) foregrounded a mainly logical and an-historical view of mathematics.

Lesson to be learned from the dilemma

The solution of Benacerraf's dilemma that I have sketched here should lead one to consider actual mathematical theories in greater detail and to take into account the historical texture of mathematics.

This does no mean turning philosophy of mathematics into history of mathematics or into philosophy of mathematical practice.
It rather means being more sensitive to the concept of presentation, in a phenomenological sense, and to "mathematical architecture" (as opposed to either a foundational or a naturalistic picture of mathematics).

Conclusion

1. An analogy can be drawn between Kant's antinomies and Benacerraf's dilemma: Dogmatism becomes the semantical account (the "standard" view), empiricism becomes the combinatorial account (the epistemic view).
   In this perspective, Benacerraf's dilemma can then be reconsidered as a mathematical antinomy about mathematics.

   The analogy turns out to be remarkably steady and sharp.

2. The main motivation of the analogy remains the prospect of transposing Kant's solution so as to suggest a way of solving Benacerraf's dilemma itself.
   The upshot of the analogy is that a mathematical object can never be given in itself, because it consists of the open-ended series of all its possible presentations (in the sense specified above).

3. The semantical and the combinatorial accounts are both wrong because they crystallise an open-ended series into either a closed referent or a complete infinite series.

4. The solution of Benacerraf's dilemma drawn from the analogy with Kant consists in acknowledging that mathematical objects do exist, but also in conceiving of each mathematical object as a series of equivalent presentations always in the making. Any such series of presentations cannot be separated from the proofs explaining that these presentations are indeed equivalent, and how they are.

5. This solution of Benacerraf's dilemma calls for a more history-sensitive analysis of mathematical objectivity.

The main conclusion

It is that a presentational, Kant-inspired solution of Benacerraf's dilemma is quite elementary and, to tell the truth, quite obvious.

Philosophy of mathematics should not remain hindered by Benacerraf-type dilemmas.

The real question is not the dilemma itself, but why it has been such central an issue for the last 40 years.

One thing is sure: It is all the more time to reconcile nominalism and platonism in philosophy of mathematics.