Structures and settings in mathematics. A presentational account of mathematical objectivity

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My talk will pursue two goals.

- First, I will show that combinatorics provides good examples of what a *mis*understanding in mathematics can be, and may thus cast some light on mathematical understanding and mathematical objectivity.
- This point comes close to the second goal of my talk, which is to reconsider the “identity problem” faced by the structuralist interpretation of mathematics.

According to *ante rem* structuralism, mathematics studies structures, conceived of as certain configurations of pure relata. Their constituents do not have any individuality, since each of them is entirely characterized (individuated/identified) by the relationships it has with all other constituents.
In other words, a mathematical object is but an item within a structure, where it can have *relational* properties only.

The essence of a natural number is its relations to other natural numbers. The subject matter of arithmetic is a single abstract structure, the pattern common to any infinite collection of objects that has a successor relation with a unique initial object and satisfies the (second-order) induction principle. The number 2, for example, is no more and no less than the second position in the natural-number structure; 6 is the sixth position. Neither of them has any independence from the structure in which they are positions, and as places in this structure, neither number is independent of the other. (Shapiro, Philosophy of mathematics. Structure and Ontology, 1997, p. 72)

Structuralism is a strong form of platonism about mathematics, since it claims structures to exist, and to exist prior both to their places and to any exemplification that they may have in the world.

> *We distinguish the office of vice president, for example, from the person who happens to hold that office in a particular year, and we distinguish the white king's bishop from a piece of marble that happens to play that role on a given chess board. [. . .] Similarly, we can distinguish an object that plays the role of 2 in an exemplification of the natural-number structure from the number itself. The number is the office, the place in the structure. The same goes for real numbers, points of Euclidean geometry, members of the set-theoretic hierarchy, and just about every object of a nonalgebraic field of mathematics.*

Shapiro cont'd:

> *Each mathematical object is a place in a particular structure. There is thus a certain priority in the status of mathematical objects. The structure is prior to mathematical objects it contains, just as any organization is prior to the offices that constitute it. The natural-number structure is prior to 2, just as "baseball defense" is prior to "shortstop" and "U.S. Government" is prior to "vice president." ([Shapiro, 1997], p. 77-78)*

A given collection of individual objects, with relations between them, is what Shapiro calls a *system*.

> *A structure is the abstract form of a system, highlighting the interrelationships among the objects, and ignoring any features of them that do not affect how they relate to other objects in the system.([Shapiro, 1997], p. 74)*

However, a structure is not the result of its abstraction from the diverse systems that instantiate it.
Ante rem structuralism is "the thesis that mathematical structures exist prior to, and independent of, any exemplifications they may have in the non-mathematical world." ([Shapiro, 2006b], p. 109)

Indeed, Shapiro distinguishes between the "*places-are-offices* perspective" and the "*places-are-objects* perspective.

- In the first one, objects are what fill the places of structures. This corresponds to *in re* structuralism: places are but offices, and do not exist apart from their particular instances. Each instanciation of the structure is its realization.
- In the second perspective, places themselves are taken to be full-fledged objects in their own right, and cannot be reduced to mere façons de parler. Each instanciation of the structure is its emanation.

The second perspective is legitimate, so claims Shapiro; It is the one endorsed by *ante rem* structuralism as a realist ontology of mathematics.

As a consequence, the places-as-objects of a given structure make up a system which instantiates itself as a structure:

> *In mathematics [. . .], the places of mathematical structures are as bona fide as any objects are. So, in a sense, each structure exemplifies itself. Its places, construed as objects, exemplify the structure. ([Shapiro, 1997], p. 89)*

Within *ante rem* structures, the most salient and maybe distinctive feature of *mathematical* structures is is that "In mathematical structures [. . .] the relations are all *formal*, or *structural*." ([Shapiro, 1997], p. 98)

In other words, a place in a mathematical structure is entirely determined by the interrelationships which are constitutive of the structure and which connect that given place to all the other places of the structure.
A place-as-object is an object but, in the case of a mathematical structure, a place-as-object is also an object-as-place, i.e., an object that boils down to the role that it plays (the place that it occupies) within the structure under consideration.

This is connected to what Shapiro calls the “freestanding” status of mathematical structures:

> *In contrast [to other kinds of structures, such as the U.S. Government], mathematical structures are freestanding. Every office is characterized completely in terms of how its occupant related to the occupants of the other offices of the structure, and any object can occupy any of its places. In the natural-number structure, for example, there is no more to holding the 6 office than being the successor of the item in the 5 office, which in turn is the successor of the item in the 4 office. ([Shapiro, 1997], p. 100)*

According to *ante rem* structuralism, a natural number is nothing else but a place in the natural-number structure, and a place in the natural-number structure is nothing else but the bundle of relations that this place (any placeholder occupying this place) has with all the other places of the structure, by virtue of the very nature of this structure.

Ante rem structuralism faces two major problems concerning identity:

- the problem of trans-structural identity
- the problem of intra-structural identity.

The problem of trans-structural identity is the problem as to whether two places belonging to two different structures can be said to be identical or different.

(Think of 1 as both a place in the natural-number structure $(0, 1, 2, \ldots)$ and a place in the integer-structure $(\ldots, -2, -1, 0, 1, 2, \ldots)$.)

Standard answer: The identity relation makes sense only within a given structure.
There is no fact of the matter as to whether two positions are the same unless they belong to the same structure.

Another possible answer is that the identity of two items belonging to two different structures makes sense in case those structures are both enbedded in a richer structure.

The problem of intra-structural identity is the problem as to whether two distinct objects of the same structure can have exactly the same essential (i.e., structural) properties and yet be distinguished.

This is the ‘ìdentity problem,” for short, as it has been brought up by Jukka Keränen in [Keränen, 2001]:
There are structures in which two distinct places are still structurally indistinguishable (despite being distinct).

Very elementary examples of that situation are the points of the geometric plane, or the conjugate complex numbers (i.e., $a + ib$ and $a - ib$).

Keränen focuses on i and $-i$ in the structure $\mathbb{C}$ of complex numbers.
Any formula in the language of $\mathbb{C}$ that is true of i is equally true of $-i$.

How to conceive of two distinct yet structurally indiscernable places of the same structure?

Any structure which admits of a nontrivial automorphism will raise the same problem.

In Keränen's view, any theory of a certain kind of objects has to provide one with individuation criteria for those objects.

The *ante rem* structuralist, however, cannot but individualize a place within a structure by the collection of relations that this place has with all the other places of the same structure.

As a consequence, any structure endowed with a nontrivial automorphism is a violation of the only identity criteria that are available to the *ante rem* structuralist, who thus cannot account for such basic mathematical domains as Euclidean geometry or complex analysis.

This is in turn a clear violation of the "fidelity constraint," according to which the philosopher of mathematics has to explain mathematical objects while remaining faithful to mathematical practice.

Another example of “symmetrical” structure comes from graph theory:

Shapiro's answer ([Shapiro, 2006b], p. 134-138) is that a formal language contains only denumerably formulae, so that it is impossible anyway to individualize all the members of nondenumerable structure, regardless of the adequacy of structuralism.

However, Shapiro points out that it is always possible to differentiate two given distinct objects. To be specific, the pair $\langle i, -i \rangle$ satisfies the formula $x + y = 0$, whereas $\langle i, i \rangle$. does not. This suffices to vindicate the fact that i and $-i$ are distinct.

La stratégie de Shapiro's strategy thus consists in substituting the task of differentiating two given objects (what Keränen calls a *condition of identity*] for a given object for the task of individuating any object (what Keränen calls an *account of identity*)

Still, according to Keränen, any theory of ontology must give a *general* account of identity, and the structuralist is bound to endorse the following general principle:
Given any system S, and given variables 'x' and 'y' ranging over the places of the structure **S** of S

$$\text{(STR)}\quad \forall x \forall y (x = y \Leftrightarrow \forall \phi (\phi \in \Phi \Rightarrow (\phi(x) \Leftrightarrow \phi(y)))),$$

where ϕ ranges over the relational properties in S that can be specified without making explicit reference to particular elements in S.

Following this account, relationally indiscernible places are identical.

Shapiro's answer simply is that there is no reason why one should expect that two distinct places of the same structure can be distinguished by at least one essential property (i.e., by the one having a purely relational property within the structure and the other lacking that same property).

Keränen again:

> *In this way [owing to (STR)], each place of* **S** *is individuated by the 'purely structural' relational properties its occupants have to the occupants of the other places of* **S***. Thus, whenever two distinct elements in S that have the same intra-systemic relational properties, according to (STR) they occupy the same place of* **S***. While this conclusion should seem prima facie plausible, in many cases it leads into absurdities. ([Keränen, 2006], p. 147-148)*

An example of the absurdities to which Keränen refers is that i and $-i$ should be recognized as identical.

There is an ambiguity in Keränen's formulation. It comes from the back and forth between a system S and its underlying structure $\mathbf{S}$.

Keränen assumes that different items of S (that is, items having different places in S) could have the same place in $\mathbf{S}$. In other words, he assumes that different places in a system could correspond to a single place in the corresponding structure.

But this assumption is of course to be excluded: The non-coincidence of two places in a structure is shown by the non-coincidence of the two items that occupy those places in a given system. This constitutes an irreducible datum.

As Shapiro puts it, "we know what identity is:"

> *For what it is worth, my view is that we understand identity (and the other logical terminology) well enough, and can use it (and them) to give implicit definitions of various mathematical structures. For example, we declare that the cardinal three structure has three distinct places. Nothing in the language of the theory distinguishes those places, but they are three, and not one, nevertheless. I remain unconvinced that a theory of a structure somehow needs an 'account of identity' in Keränen's sense. We know what identity is. ([Shapiro, 2006a], p. 171)*

Two items being relationally indiscernibles in a system S which instantiates a structure **S** *only* means that the system S' obtained by swapping one for the other is still a system that instantiates **S**: The two items do not need to be identified. Shapiro is right in that.

On the other hand, Keränen is equally right in that the realist structuralist has no choice but to acknowledge that two relationally indiscernibles places within the one and same structure are identical.

There is a privileged domain in mathematics which calls for its consideration —a domain where permutations and symmetries are brought to the fore, namely combinatorics.

Studying the very basic framework of combinatorics should give one the insight needed to understand how automorphisms and symmetries work, and thus how to reconsider the Keränen *vs* Shapiro controversy.

This will allow us to develop the second thread I would like to present: misunderstanding in mathematics.

Suppose that three items a, b, c are given, and let's consider the permutation that swap a and b while leaving c unchanged.
This is a very elementary mathematical scenario. But is it so simple a thing to grasp?

The set composed of a, b and c is not altered by the permutation under consideration: This permutation is, so to speak, a mere fancy of the mind.

Yet we cannot help but picture a, b and c spatially, assigning to those objects different respective positions.
Doing that, we really could have put originally a at the position that is actually occupied by b, and put originally b at a's actual position.

So how to conceive of the permutation, since it is supposed to be carried it out *with respect to a setting which is itself defined only up to any arbitrary permutation of the original positions of* a, b *and* c?

It is really something difficult to get one's head around.

A permutation is a one-to-one mapping of a set to itself. One usually writes a permutation by using schematic letters. For instance, the permutation on a three-object set which exchanges the two first ones and leaves the third one untouched is usually written:

$$\left(\begin{array}{ccc} a & b & c \\ b & a & c \end{array}\right).$$

Now, it is obvious that the choice of $\{a, b, c\}$ instead, say, $\{\alpha, \beta, \gamma\}$ or $\{a_1, a_2, a_3\}$, is completely immaterial.

This does not mean, however, that one is considering the set $\{a, b, c\}$ up to a permutation replacing a with α, b with β, and c with γ. Indeed, keeping track of permutations precisely presupposes that letters have been settled once and for all.

The letters a, b and c are variable parameters to the extent that they are arbitrary, but —of course— they are not variable in the sense of the variation that they make it possible to represent.

Threefold arbitrariness attached to any representation of a permutation:

- the arbitrariness of the underlying set:

$$\begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix} \equiv \begin{pmatrix} \alpha & \beta & \gamma \\ \beta & \alpha & \gamma \end{pmatrix}.$$

- the arbitrariness of the original arrangement of the members:

$$\begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix} = \begin{pmatrix} b & a & c \\ a & b & c \end{pmatrix}.$$

- the arbitrariness of the labelling of the members of that set:

$$\begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix} \sim \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix}.$$

I am mainly interested in the the arbitrariness of the labelling.

It looks like what the first permutation does with b is what the other does with c, and vice versa. It is as if both permutations were doing exactly the same thing, except that the labels of b and c have been swapped. And well, after all the object called 'b' could have been called 'c', and conversely.

In reality, it makes absolutely no sense to mark off the objects from their names. We do not have access to a otherwise than by its name. The very distinction between objects and names is confused.

Each of the mathematical objects that combinatorics deals with *is* its own label.
See Camille Jordan, Livre II of his *Traité des substitutions et des équations algébriques*:

> *On donne le nom de substitution à l'opération par laquelle on intervertit un certain nombre de choses que l'on peut supposer représentées par des lettres* $a, b, \ldots$.

As to the arbitrariness of the labelling, the permutations

$$\sigma = \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix} \text{ and } \sigma' = \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix}$$

are said to be *conjugate*.

Quite generally, two permutations σ et σ' on a set E are *conjugate* if there exists a third permutation τ on E such that $\sigma' \circ \tau = \tau \circ \sigma$.

Conjugacy is an equivalence relation, and it appears that the arbitrary choice of a labelling is nothing else but the arbitrary choice of a representant within a conjugacy class.

The representation of a permutation involves arbitrary choices which can themselves be cashed out (mathematically controlled) in terms of permutations.
A permutation is mathematically an invariant under typographical permutations.

So, to understand how to represent a permutation, one ought to have some prior understanding of what a permutation is.

There is no vicious circle here, still things turn out to be trickier than one could expect them to be.

The set (group) of all permutations on a set X with n elements is written $\mathfrak{S}_n$.
Strictly speaking, $\mathfrak{S}_n$ is defined as the group of all permutations on the particular set $\{1, 2, \ldots, n\}$.
The identification is natural, since the group of all permutations $\mathfrak{S}_X$ on *any* n-element set X is isomorphic to $\mathfrak{S}_n$.

The isomorphism, however, is not canonical. And selecting one precisely amounts to selecting a certain labelling (or numbering) of X.

In the case $X = \{1, 2, 3, \ldots, n\}$, the numbering is trivial, i.e., amounts to taking each number as its own numeral:
$1 = 1, 2 = 2, \ldots, n = n$.

So, in the usual representation of a permutation, the labels are the numerals directly corresponding to the numbers.

What happens if numbers are confused with numerals?

It is not ununderstandable to understand numerals 1, 2, 3, . . . as indices of their respective canonical ranks, rather than the underlying individual objects that these numerals are supposed to stand for throughout the permutation.

Let Π the following permutation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}.$$

Here 1, 2, 3 et 4 are not ranks, but individual objects, whose orbits are considered as Π is iterated:

$$\begin{matrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 2 & 1 \\ 1 & 2 & 3 & 4 \end{matrix}\quad .$$

Now let's imagine the following heterodox interpretation, let's call it Dummy's interpretation. Dummy understands $1, 2, 3, 4$ as ranks rather than as objects, and accordingly carries out the iteration of Π in this way:

$$R_1 = 1 \;\; 2 \;\; 3 \;\; 4 = 1.1 \;\; 2.1 \;\; 3.1 \;\; 4.1$$

$$R_2 = 4 \;\; 3 \;\; 1 \;\; 2 = 1.2 \;\; 2.2 \;\; 3.2 \;\; 4.2 \;\; .$$

The second row R_2 is understood as reindexing the ranks: 4 becomes the "new" 1, so to speak, that is, the "new" number 1 position. Similarly 3 becomes the "new" 2, and so on. In this perspective, 1 is not only 1, but first and foremost the index of the first position in the current row, and the permutation is viewed as a perturbation of the ranks in reference to $\{1, 2, \ldots, n\}$.

$$\Pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}$$

According to Dummy's reading, the calculation of the third row R_3 goes that way:

> "4 *becomes* 2*, but since* 2 *now is* 3 *(because* $2.2 = 3$*),* 4 *finally lands on* 3*. So* 3 *will be the first numeral on* R_3*.*"

The basic principle of that heterodox interpretation is that the 4 on the first row R_1 becomes 2 on the second row R_2, but 2 as understood precisely according to the new code set by R_2, namely as $2.2 = 3$. In the same way, 3 becomes 1 with $1.2 = 4$.
One finally gets:

$$R_3 \quad = \quad 3 \;\; 4 \;\; 2 \;\; 1 \quad = \quad 1.3 \;\; 2.3 \;\; 3.3 \;\; 4.3 \quad ,$$

whereas the correct iteration of Π of course is:

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \\ 2 & 1 & 4 & 3 \end{array} \quad .$$

In other words, Dummy takes the numbers on which the permutation acts to be *floating ranks*, whose counterpart is reset at each step.

Obviously, Dummy's heterodox interpretation is wrong, based on a conflation of numbers as objects (as when one says that 4 "becomes" 2) and numbers as contextual indices of positions in a given row (as when one says that 3 has become the new 2 in the context of R_2).

Dummy's interpretation is wrong, but that is beside the point, which is that this interpretation is not as such nonsensical. This is a case of misunderstanding about mathematics of which one can find some kind of mathematical explanation.

Let's give another example of Dummy's misinterpretation. The correct iteration of the cycle $(12345) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$ is:

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{array} .$$

The first three rows, in Dummy's version, are on the contrary:

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 4 & 5 & 1 & 2 & 3 \end{array} .$$

Indeed, the first item in the third row is not 3, but 3 as a rank, i.e., the holder of the third position as determined in the second row, namely 4.

The full iteration of (12345), in Dummy's version, is:

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 4 & 5 & 1 & 2 & 3 \\ 4 & 5 & 1 & 2 & 3 \\ 2 & 3 & 4 & 5 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{array} .$$

The deeper point is this: **Mathematical understanding relies on a correct understanding of how mathematical labelling works.**

Labellings, numberings, parameterizations in general turn out to be pervasive throughout mathematics, as devices to "set ideas."
I will use the general term "settings" to refer to them.

Settings are actually so pervasive and so important to grasp that any correct account of mathematical objectivity has to mention them.

Emergency plan:

- Shapiro and Keränen do not recognize settings as the fundamental intermediate level between structures and systems, hence their endless controversy.
- Settings fall to the category of *presentations*, both in the mathematical and philosophical sense.
- Hypothesis: There is a way to understand in a unified way presentations in the mathematical sense (in the sense of the presentation of a group, in particular) and presentations in the intensional or phenomenological sense (Frege's *Art des Gegebenseins*, Brentano's *Modus des Verstellens*, Husserl's *Vorstellungen*).

Nuemrous examples of settings in mathematics.
Among them, the choice of an origin and coordinate axes in geometry (shift from a vector space to an affine space).

Each setting involves arbitrary choices, and so in this sense is "variable" to the extent that it could have been different.
But this variability is only virtual: Once set, a setting cannot change. Otherwise confusions follow.

A setting is what endows a structure with a certain angle, so that the structure becomes *rigid* (i.e., deprived of any nontrivial automorphism). A setting kills all pre-existing symmetries.

Back to the Keränen / Shapiro controversy.

A setting is neither the structure itself, just as it is, nor a mere particular system instantiating the structure.

Shapiro thinks that a setting is a structure, so that the identity problem cannot even be raised: We can see that distinct items are distinct, "We know what identity is."

Keränen thinks that a setting is a system, so that shifting to the structure requires to get rid of any means to distinguish numerically distinct items.
This makes necessary to be able to know what identity within a particular structure amounts to in general.

A setting is a structure as soon as it is fixed, and so ceases to be noncanonical (it is given, so it ceases to be arbitrarily chosen). But otherwise it is not a structure. The labelling of the members of a given system is not relevant to study the formal relations that hold between them.

Still, a setting is not a mere system, because two different settings of the same structure do not differ as systems.

So now what does the disagreement consist in?
I think that it is this:

- Shapiro thinks that presenting a structure through a setting of it allows us to get to the structure itself. A structure contains in itself the differentiation of its objects (places).
- In Keränen's reckoning, on the contrary, a setting cannot give but itself, so that the need of a setting to reach a structure simply proves that the intended structure does not exist (as a structure in the realist-structuralist sense), because, were we able to reach it, it would collapse, like the structure $\langle Z, + \rangle$ when $1 = -1$.

To Shapiro, a setting "enriches" a structure, but as being part of it, and does not add anything.

In Keränen's view, a structure endowed with a setting cannot be but a system, an instance of the structure.

The mistake shared by both Shapiro and Keränen is to fail to distinguish between structures, systems *and settings*.

A nontrivial automorphism of a structure is always an isomorphism between two settings associated to the structure.
This solves the identity problem, but at the cost of acknowledging that a structure is never accessible "as such."

Mathematical structures can never be given "in themselves," but only through *modes of presentation*, of which settings are a fundamental component.

Settings are instrumental to setting ideas. They are parameterizing devices without whose bias the intended structure cannot be reached, i.e. cognitively handled.
This is shown by our example of the symmetric group $\mathfrak{S}_3$.

Bringing up the notion of mathematical"setting" or of mathematical "presentation" is a general way to single out the pervasive use of such devices throughout mathematics.

Settings and presentations do not boil down to sub-mathematical conditions of concrete mathematical activity, such as the actual drawing compared to the geometrical theorem. They are truly mathematical in nature.

About presentations in mathematics:
$[a, a^n = 1]$ IS (a presentation of) the cyclic group $\mathbb{Z}/n\mathbb{Z}$.

About the very phrase "mode of presentation:" This term was mentioned by Frege in the context of the distinction that he drew between sense and denotation.

The choice that Frege made of the term "presentation" could certainly be driven back to two opposite sources:

- the work of Franz Brentano in psychology, in particular his book *Psychology from an Empirical Standpoint* (1874), where the notion of "mode of presentation" (*Modus des Verstellens*) is ubiquitous
- the foundation of the theory of group presentations by Walther von Dyck (a student of Felix Klein) in his article "Gruppentheoretische Studien," published in 1882 in the *Mathematische Annalen*.

Brentano: "Every intentional experience is either a presentation (*Vorstellung*) or is founded upon a presentation."

Husserl, *Fifth Logical Investigation*.
Husserl objects to the idea that, underlying every consciousness, there are acts of a special kind which merely present objects as the content for other, higher-order acts of consciousness.

Versatility of the concept of presentation: To be explored.

Thank you !

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