Homotopy Model Theory

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Homotopy theory dramatically entered the scene of logic through the connections that have been made between Martin-Löf type theory and model categories.
This is Homotopy Type Theory.

I would like to show that logic can be connected with homotopy theory through model theory.
That would be Homotopy Model Theory.

Starting point

The notion of boundary can easily be transposed in the context of first-order logic, formulae being conceived of as chains (in the sense of a formal sum of faces and, in homology, in the sense of a chain complex).

The boundary of a given formula $\phi(v_0, v_1, \ldots, v_n)$ with exactly $v_0, v_1, \ldots, v_n$ as free variables, can be defined as follows:

$$\partial\phi := \bigwedge_{i=0}^{n-1} \neg^i \forall x \phi(v_0, \ldots, v_{i-1}, x, v_{i+1}, \ldots, v_{n-1}).$$

More specifically

Let's consider a fixed first-order language L with equality

- which contains at least a unary quantifier Q
- whose free variable symbols are exactly 'v_i', $i \geq 0$
- and whose bound variable symbols are exactly 'x', 'y', 'z', and so on.

Formulae of L will be taken up to bound variable renaming, but not up to free variable renaming.

Formulae will be so written that any variable with n free variables has exactly $v_0, v_1, \ldots, v_{n-1}$ as free variables.
Otherwise put, an L-formula (in the usual sense) such as $(P(v_0) \wedge \neg P(v_2))$ will not be deemed to be a well-formed formula.

Formulas as chains

Let's get back to

$$\partial\phi := \bigwedge_{i=0}^{n-1} \neg^i \forall x \phi(v_0, \ldots, v_{i-1}, x, v_{i+1}, \ldots, v_{n-1})$$

and let's consider, for instance, a formula $\phi(v_0, v_1, v_2)$ with exactly three free variables v_0, v_1 and v_2.

One gets, successively:

- first, the conjunction of $\forall x \phi(x, v_0, v_1)$, $\neg\forall x \phi(v_0, x, v_1)$ and $\forall x \phi(v_0, v_1, x)$;
- then, the inconsistent conjunction of the following six formulae: $\forall y \forall x \phi(x, y, v_0)$ and $\neg\forall y \forall x \phi(x, v_0, y)$, $\forall y \neg\forall x \phi(y, x, v_0)$ and $\neg\forall y \neg\forall x \phi(v_0, x, y)$, $\forall y \forall x \phi(y, v_0, x)$ and $\neg\forall y \forall x \phi(v_0, y, x)$.

So in the end:

$$\partial(\partial\phi) \equiv \bot$$

Simplicial ideas

Let F_n be the set of formulae of L with exactly $v_0, \ldots, v_n$ as free variables.
(F_{-1} may be defined as the set of all sentences of L.)

The two following applications $d_i : F_n \to F_{n-1}$ and $s_i : F_n \to F_{n+1}$ can then be defined:

$$d_i(\phi(v_0, \ldots, v_n)) = Qx\phi(v_0, \ldots, v_{i-1}, x, v_i, \ldots, v_{n-1})$$

$$s_j(\phi(v_0, \ldots, v_n)) = ((v_j = v_{j+1}) \to \phi(v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_{n+1})).$$

Provided that the quantifier Q satisfies, for every formula φ

(a) $QyQx\varphi(y, x, \vec{u}) \equiv QyQx\varphi(x, y, \vec{u})$

(b) $Qx((x = y) \rightarrow \varphi(y, \vec{u})) \equiv \varphi(x, \vec{u})$,

the following equalities (up to logical equivalence) are verified:

- $d_i d_j = d_{j-1} d_i$ for $i < j$
- $s_i s_j = s_{j+1} s_i$ for $i < j$
- $d_i s_j = \begin{cases} s_{j-1} d_i & \text{for } i < j \\ \text{id} & \text{for } i = j \text{ or } i = j + 1 \\ s_j d_{i-1} & \text{for } i > j + 1 \end{cases}$

Definition

A simplicial set is a sequence $(X_n)_{n\geq 0}$ of sets, together with maps $d_i^n : X_n \to X_{n-1}$ $(0 \leq i \leq n)$ and $s_j^n : X_n \to X_{n+1}$ $(0 \leq j \leq n)$, for each n, satisfying the simplicial identities:

$$\begin{cases} d_i d_j = d_{j-1} d_i & \text{if } i < j \\ d_i s_j = s_{j-1} d_i & \text{if } i < j \\ d_j s_j = d_{j+1} s_j = \mathrm{id} \\ d_i s_j = s_j d_{i-1} & \text{if } i > j+1 \\ s_i s_j = s_{j+1} s_i & \text{if } i \leq j \end{cases}$$

Proposition

$F_*^Q = \langle F_n, (d_i^n)_{0\leq i\leq n}, (s_j^n)_{0\leq j\leq n}\rangle_{n\in\mathbb{N}}$ is a simplicial set.

In this perspective ("formulae as chains"):

- Any (generalized unary) quantifier Q satisfying conditions (a) and (b) becomes a "face operator" (i.e., the sequence (d_i^Q))
- while (s_j) appears to be the corresponding sequence of "degeneracy operators."

What about connectives?

Introducing bisimplicial objects, we are in a position to characterize the usual connectives.
Indeed, the commutative diagram:

$$\begin{array}{ccc} F_{m,p} & \xrightarrow{\;c\;} & F_n \\ {\scriptstyle \langle Q,Q' \rangle}\downarrow & & \downarrow{\scriptstyle Q''} \\ F_{m-1,p-1} & \xrightarrow[\;c'\;]{} & F_{n-1} \end{array}$$

is a way to express $Q''x(\phi c \psi) \equiv (Qv_i\phi)c'(Q'v_i\psi)$ for any formulae $\phi \in F_m$, $\psi \in F_p$ (here $n = \max(m, p)$).

We have that $\wedge$ is characterized by:

$$\begin{array}{ccc} F_{m,p} & \xrightarrow{c} & F_n \\ \big\downarrow{\scriptstyle \langle\forall,\forall\rangle} & & \big\downarrow{\scriptstyle \forall} \\ F_{m-1,p-1} & \xrightarrow[c]{} & F_{n-1} \end{array}$$

and $\vee$ by:

$$\begin{array}{ccc} F_{m,p} & \xrightarrow{c'} & F_n \\ \big\downarrow{\scriptstyle \langle\exists,\exists\rangle} & & \big\downarrow{\scriptstyle \exists} \\ F_{m-1,p-1} & \xrightarrow[c']{} & F_{n-1} \end{array}$$

Negation becomes a simplicial morphism between $F_*^{\forall}$ and $F_*^{\exists}$. Indeed,

$$\begin{array}{ccc} F_n & \xrightarrow{\neg} & F_n \\ \downarrow{\scriptstyle\forall} & & \downarrow{\scriptstyle\exists} \\ F_{n-1} & \xrightarrow[\neg]{} & F_{n-1} \end{array}$$

commutes, and in fact negation is characterized by that condition.

Simplicial topology

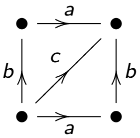
CW complexes were introduced in algebraic topology in order to analyze "nice" topological spaces as reconstructible from elementary "cells," as the result of glueing cells along other cells of smaller dimension.

A simplicial complex X is just a recipe for joining polyhedra together so as to obtain a CW complex $|X|$, called the topological realization of X.

Conversely, given a CW complex X, a simplicial complex whose realization is (up to homeomorphism) identical with X can be conceived of as a triangulation of X.

Simplicial complexes can be described diagrammatically.

For instance,



is the representation of the two-dimensional torus $\mathbb{T}^2$.

(The tags on arrows indicate how to glue edges together and thus reconstruct the torus as if through a paper folding exercise.)

Simplicial sets were introduced as a means to encode the purely combinatorial properties of the construction and shape of CW complexes.
They retain the mere skeleton of the building pattern of CW complexes and are deprived of any topology, in contrast with CW complexes.

Yet, quite surprisingly, simplicial sets are sufficient to capture most features relevant to homotopy theory.
Simplicial topology is the conceptual core of modern homotopy theory.

Owing to both their combinatorial nature and their topological meaning, their application to logic should come as no surprise, and supply an interesting connection between the combinatorial aspect of logical syntax and the use of topological methods in model theory.

Semantics

Let M an L-structure.

$(\varphi(v_0, v_1))^M \subseteq |M|^2$

$(\exists x \varphi(v_0, x))^M \subseteq |M|$

Definition

Let T be a fixed L-theory. Given a model M of T,

$$M_* := F_*^{\exists,M} = \langle D_n(M), (\exists_i^{n,M})_{0\leq i\leq n}, (s_j^{n,M})_{0\leq j\leq n}\rangle_{n\in\mathbb{N}}$$

where:

- $D_n(M)$ (for $n \geq 0$) is the set $\mathrm{Def}_{n+1}(M)$ of all definable subsets of $|M|^{n+1}$ and $D_{-1}(M)$ is the theory $Th(M)$ of M
- $\exists_i^{n,M} : D_n(M) \to D_{n-1}(M)),\ A = \{\vec{a} \in |M|^{n+1} : M \vDash \phi_A(v_0, \ldots, v_n)[\vec{a}]\} \mapsto \{\vec{a'} \in |M|^n : M \vDash \exists x\ \phi_A(v_0, \ldots, v_{i-1}, x, v_i, \ldots, v_{n-1})[\vec{a'}]\}$ are the face operators
- $s_j^{n,M} : D_n(M) \to D_{n+1}(M),\ A \mapsto \{(\vec{x}, y) : \vec{x} \in A \text{ and } y = x_j\}$ are the degeneracy operators.

Proposition

For any M, M_* is a simplicial set.

General remark

In general, a model cannot be reconstructed from the hierarchy of its definable subsets. Two models with the same definable subsets (up to isomorphism) need not be elementarily equivalent.

Conversely, any two elementary equivalent L-structures M and N have isomorphic hiearchies of definable subsets: For any L-formula ϕ, it suffices to take $\phi^M \mapsto \phi^N$.
Yet this latter isomorphism does not come from any actual map between M and N.

Basic notions of model theory: substructure (embedding)

and elementary substructure (elementary embedding).

Tarski-Vaught test:
M being a substructure of N, M is an elementary substructure of N iff, for any formula $\phi(x, a_1, \ldots, a_n)$ with parameters $a_1, \ldots, a_n$ from M,
$N \models \exists x \phi(x, a_1, \ldots, a_n)$ implies that there is $a \in |M|$ such that $N \models \phi(x, a_1, \ldots, a_n)[a]$.

Definition

Given two simplicial sets X and Y, a simplicial map $f : X \to Y$ is a family of maps $f_n : X_n \to Y_n$ $(n \geq 0)$ which commute with the operators d_i and s_j, e.g.:

$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & Y_n \\ {\scriptstyle d_i^X}\downarrow & & \downarrow{\scriptstyle d_i^Y} \\ X_{n-1} & \xrightarrow[f_{n-1}]{} & Y_{n-1}. \end{array}$$

Definition

Let M be a substructure of an L-structure N. For each $n \geq 0$, the restriction map r_n sends ϕ^N to $\phi^N \cap |M|^{n+1}$, for each formula $\phi(v_0, \ldots, v_n) \in F_n$.

Theorem

A substructure M of an L-structure N is an elementary substructure of N iff the restriction maps r_n induce a well-defined simplicial map $r_* : N_* \to M_*$.

Remark: The restriction maps r_n making up a simplicial map from N_* to M_* is the direct expression that the Tarski-Vaught test is met.

$$\begin{array}{ccc} D_n(N) & \xrightarrow{r_n} & D_n(M) \\ \downarrow{\scriptstyle \exists_i^{n,N}} & & \downarrow{\scriptstyle \exists_i^{n,M}} \\ D_{n-1}(N) & \xrightarrow[r_{n-1}]{} & D_{n-1}(M) \end{array}$$

Corollary

The mapping $(-)_*$ is a contravariant functor from the category of L-structures and elementary embeddings, to the category of simplicial sets and simplicial maps.

Each elementary embedding $f : M \to N$ gives rise to $f_* : N_* \to M_*$ in a functorial way, with
$f_n : \phi^N \mapsto \{\vec{a} \in |M|^{n+1} : N \models \phi[f(\vec{a})]\} = \phi^M$.

Let M be an elementary substructure of N, and let's suppose that $|M|$ is definable in N, by a formula $\underline{M}(x)$ of L. For any $\phi \in F_n$, $\phi^{\underline{M}}$ is the formula $(\underline{M}(v_0) \wedge \ldots \wedge \underline{M}(v_n) \wedge \phi^{(\underline{M})})$, where $\phi^{(\underline{M})}$ means the relativization of ϕ to $\underline{M}$. This relativization allows one to define extensions $e_n : \phi^M \in D_n(M) \mapsto (\phi^{\underline{M}})^N \in D_n(N)$, and thus a morphism $e_* : M_* \to N_*$.

Definition

Given two simplicial sets X and Y, a *retraction* of Y over X consists of a pair $\langle f, g\rangle$ of simplicial morphisms $f : X \to Y$ and $g : Y \to X$, such that $g \circ f = \mathrm{id}_X$ (which implies that f is a monomorphism).

Theorem

Let M be an elementary substructure of N. Then the domain $|M|$ of M is definable in N iff $\langle e_*, r_*\rangle$ defines a retraction of N_* over M_*.

Types

Definition

Given a complete theory T in L (for instance, the theory $T(M)$ of some L-structure M), an *n-type* is a set of formulae with exactly $v_0, \ldots, v_n$ as free variables, which is consistent with T.
The set of all n-types is written $S_n(T)$.

Proposition

$S_*(T) = \langle S_n(T), (\exists_i^n)_{0 \le i \le n}, (s_j^n)_{0 \le j \le n} \rangle_{n \in \mathbb{N}}$
('$\exists_i$' being just a shorthand for $\exists v_i$)
is a simplicial set.

(There is a connection already established here with topology. Indeed, each $S_n(T)$ can be endowed with a "Zariski-type" topology. It could be interesting to pursue this connection the framework of simplicial sets.)

Definition

Let $p \in S_1(M)$ be a 1-type over some saturated model M.
The type p is said to be definable iff to each formula $\phi(v_0, v_1, \ldots, v_n)$ corresponds a formula $d_i^{p,n}(\phi)(v_0, \ldots, v_{n-1})$ such that, for any tuple $(a_1, \ldots, a_n) \in |M|^n$, $\phi(a_1, \ldots, a_{i-1}, v_0, a_i, \ldots, a_n) \in p$ iff $M \vDash d_i^{p,n}(\phi)(v_0, \ldots, v_{n-1})[\vec{a}]$.
The operator d^p is called the *definition* of p.

One has that $d_i^{p,n}(\neg\phi) = \neg d_i^{p,n}(\phi)$ and that $d_i^{p,n}(\phi \wedge \psi) = d_i^{p,n}(\phi) \wedge d_i^{p,n}(\psi)$.
Besides, each F_n can be turned into a group, with $\nleftrightarrow$ as the binary law, and so each $d_i^{p,n}$ is a group homomorphism.
Consequently

$$F_*^p := \langle\langle F_n, \nleftrightarrow, \perp\rangle, (d_i^{p,n})_{0\leq i\leq n}, (s_j^n)_{0\leq j\leq n}\rangle_{n\in\mathbb{N}}$$

is a simplicial group.

Furthermore, writing '$\odot$' for '$\not\leftrightarrow$'

$$\partial^p(\phi) := \bigodot_{i=0}^{n} d_i^{p,n}(\phi)$$

(recall the definition of $d_i^{p,n}$: For $\phi \in F_n$,
$\phi(a_1, \ldots, a_{i-1}, v_0, a_i, \ldots, a_n) \in p$ iff $M \models d_i^{p,n}(\phi)(v_0, \ldots, v_{n-1})[\vec{a}]$),

one gets a differentiation operator:

$$\partial^p \circ \partial^p \equiv \bot$$

This means that $\langle F_*^p, \partial^p \rangle$ is a chain complex.

Realization

Standard n-simplex:

$$|\Delta^n| = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} : t_i \geq 0, \sum_{i=0}^{n} t_i = 1\}.$$

Hence, $|\Delta^1|$ is the segment $[0, 1]$, $|\Delta^2|$ the full triangle (the whole triangular surface), $|\Delta^3|$ the 3-dimensional analog in $\mathbb{R}^4$, and so forth.
The realization of a simplicial set X is the topological space $|X|$:

$$|X| = \varinjlim_{\Delta^n \to X} |\Delta^n| = (\coprod_{n \geq 0} X_n \times |\Delta^n|)/\sim,$$

where $\sim$ expresses the identification of parts that are glued together.
An element of $|X|$ is of the form $\overline{(x, u)}$ with $x \in X_n$ and $u = (t_0, \dots, t_n)$.

Model-theoretic Functors

Let $L = \{R_k^{(n)} : n, k \in \mathbb{N}\}$ a fixed first-order language, endowed with a fixed Gödel numbering. For any L-formula ϕ, $[\phi]$ denotes the number assigned to ϕ according to that numbering.

Definition

Let **L-Strs** be the category of all L-structures and elementary embeddings.

For any object M of **L-Strs**, M_* is a simplicial set, and each M_n is linearly ordered as follows:

For any $D \in D_n(M)$, let $[D]$ be, among all the formulae defining D (that is, coextensional over M), the number of the formula whose number is the least.

Definition

A *linearly ordered simplicial set* is a simplicial object in the category of linear orders and linear order preserving maps between them.

Proposition

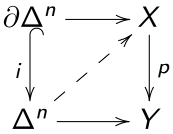
For any L-structure M, M_* is a linearly ordered simplicial set.

The n-th standard simplicial set is $\Delta^n = \mathrm{Hom}_\Delta(-, n)$.

$\partial\Delta^n$ = boundary of Δ^n.
(Think of Δ^n through its realization, namely the standard n-simplex $|\Delta^n|$.)

Definition

A *trivial Kan fibration* is a map $p : X \to Y$ of simplicial sets satisfying the following *lifting condition*:
For every commutative square of simplicial morphisms



,

there is a map $\theta : \Delta^n \to X$ making the whole diagram commute.

(This definition can be rephrased in a combinatorial way.)

Proposition

For any elementary embedding $f : M \xrightarrow{\prec} N$ *between* L*-structures,* f_* *is a trivial Kan fibration.*

In other words, the Tarski-Vaught test can be related to a lifting condition (in the sense of the definition of a Kan fibration).

Definition

Let **losSets** be the category whose objects are linearly ordered simplicial sets and whose arrows are all linear order preserving trivial Kan fibrations (i.e., an arrow in **losSets** is a trivial Kan fibration as a map of linearly ordered simplicial sets).

Corollary

$F = (-)_*$ *is a functor from* $\mathbf{L\text{-}Strs^{op}}$ *to* **losSets**.

Conversely, given any linearly ordered simplicial set X, there is an L-structure $\widehat{X}$ associated to it:

Definition

Let X be a linearly ordered simplicial set. One then defines:

- $|\widehat{X}|$ is $|X|$, the geometric realization of X;
- $|X|_n^k$ is the image of X_n^k, the k-th n-simplex of X, in $|X|$;
- s_n^k is the first vertex of $|X|_n^k$;
- $(R_k^{(n)})^{\widehat{X}}$ is the set of all n-uples $(x_0, \ldots, x_{n-1}) \in |X|^n$ such that the convex hull of $(s_n^k, x_0, \ldots, x_{n-1})$ is an homeomorphic subset of $|X|_n^k$.
  $((R_k^{(n)})^{\widehat{X}} = \emptyset$ if X_n^k contains less than k simplices.

Definition

Let $p : X \to Y$ be a linear order preserving trivial Kan fibration. One then defines $\widehat{p} : \widehat{Y} \to \widehat{X}$ by:

$$\widehat{p}(\overline{(y,u)}) = \overline{(p_n^{-1}(y),u)}.$$

Proposition

For any linear order preserving trivial Kan fibration $p : X \to Y$, $\widehat{p} : \widehat{Y} \to \widehat{X}$ is an elementary embedding.

Corollary

$G = \widehat{(-)}$ is a functor from $\mathbf{losSets^{op}}$ *to* $\mathbf{L\text{-}Strs}$.

Proposition

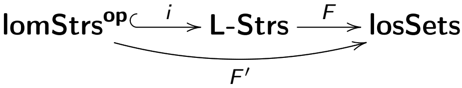
For any linearly ordered simplicial set X, $\widehat{X}$ is an o-minimal L-structure.

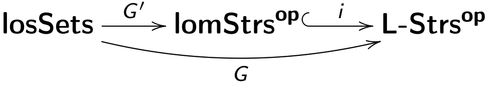
Definition

Let **LomStrs** the full subcategory of **L-Strs** composed of all o-minimal L-structures.

Corollary

There are two functors F' : $\mathbf{lomStrs^{op}} \to \mathbf{losSets}$ *and* G' : $\mathbf{losSets} \to \mathbf{lomStrs^{op}}$ *such that:*





Open Question

Are F' and G' adjoint functors?

Conclusion: Guideline analogy

between elementary equivalence and homotopy equivalence.

Two L-structures are elementary equivalent iff they have isomorphic ultrapowers.

Two topological spaces are homotopy equivalent iff they are both deformation retracts of a single larger space.