# A Stack of Sets

Brice Halimi

Université Paris Ouest Nanterre La Défense (Philosophie)
& SPHERE

My first talk was about fibered structures for modal logic.

Pursuing the application of fibered structures to logic, today's talk is about fibered structures for set theory.

More specifically, I will consider Algebraic Set Theory (AST), which shows a very interesting example of positive interaction between (ZF-style) set theory and category theory.

I will focus on the first three axioms of the original axiomatization of AST by André Joyal and Ieke Moerdijk.

I will argue that AST can be described as a genuine graft of the theory of fibered categories onto ZF, which goes beyond the unfruitful opposition often established between set theory and category theory.

It is better to try to go beyond the foundational contest between Set Theory and Category Theory.

# Presentation of AST

Two different approaches : a logical one (Alex Simpson, Steve Awodey), a geometrical one (Joyal-Moerdijk).

Joyal & Moerdijk: generalization of models of ZF ("free ZF-algebras").

Awodey & Simpson: general program of completeness results between different axiomatizations of set theory and different collections of "categories of classes."

Let $C$ be a Heyting pretopos, which means that $C$ is rich enough to interpret first-order logic and arithmetic. (Still, $C$ is not supposed to be a topos, and in particular to have "power objects.")

The basic idea is to characterize a special subcollection of the collection of all maps of $C$: a class $S$ of "small maps" in $C$.

Intuitively, **an arrow is a small map if all its fibers have a set-like size**.

Then, an object $X$ of $C$ is said to be **small** if $X \to 1$ is a small arrow.

Upshot: Arrows are brought to the fore, instead of sets, and sets themselves are conceived of as "fibers."

## Axioms for the class $S$:

1. Any isomorphism of $C$ belongs to $S$ and $S$ is closed under composition.
2. Stability under change base: for any pullback

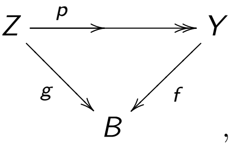
   $$\begin{array}{ccc} Y' & \longrightarrow & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow[p]{} & X \end{array} \quad ,$$

   if $f$ belongs to $S$, so does $g$.
3. "Descent": For any pullback along an epimorphic arrow $p$

   $$\begin{array}{ccc} Y' & \longrightarrow & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \overset{}{\underset{p}{\twoheadrightarrow}} & X \end{array} \quad ,$$

   if $g$ belongs to $S$, so does $f$.

4. The arrows $0 \to 1$ and $1 + 1 \to 1$ belong to $S$.
   This axiom ensures that the empty set (initial object) is small and that any finite set is small.
5. If two arrows $f : Y \to X$ and $f' : Y' \to X'$ belong to $S$, then so does their sum $f + f' : Y + Y' \to X + X'$.
   This axiom ensures that the disjoint union of two small sets (objets) is a small set.
6. In any commutative diagram




,

   if $g$ belong to $S$, then so does $f$.

Three other axioms ("Collection," "Exponentiability" and "Representability") must be added to define a class of small maps in $C$.

In the framework of the axioms for a class of small maps, Joyal and Moerdijk introduce a new way of building and understanding models of ZF, but I will not focus on that part of AST.

I will focus on the axioms 1-3 only, in particular on the Descent Axiom, and discuss the Representability Axiom briefly toward the end.

# Outline of my talk

The main purpose of my talk is to explain the elliptic phrase "Descent" used to describe the third axiom of AST.

Descent theory is a central tool of abstract algebraic geometry, coming from Grothendieck's work.

Descent theory makes sense only in the general context of "fibered categories," a.k.a. "fibrations."

So I will explain:

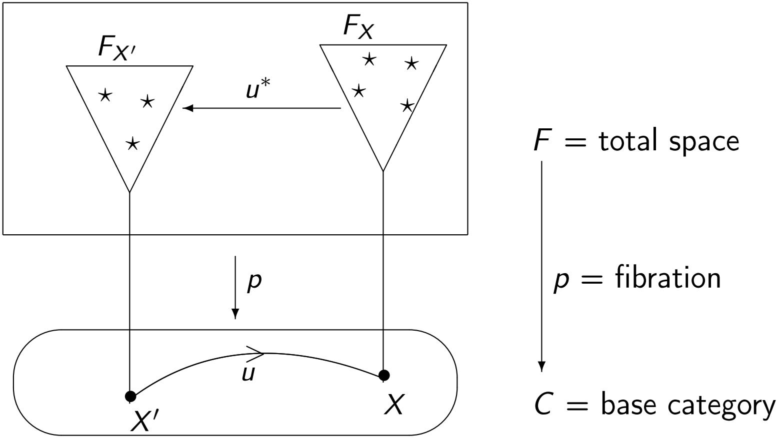
- what fibrations are;
- what descent consists in;
- the rationale of the first three axioms of AST;
- three extensions of the perspective laid out by AST.

# Fibrations

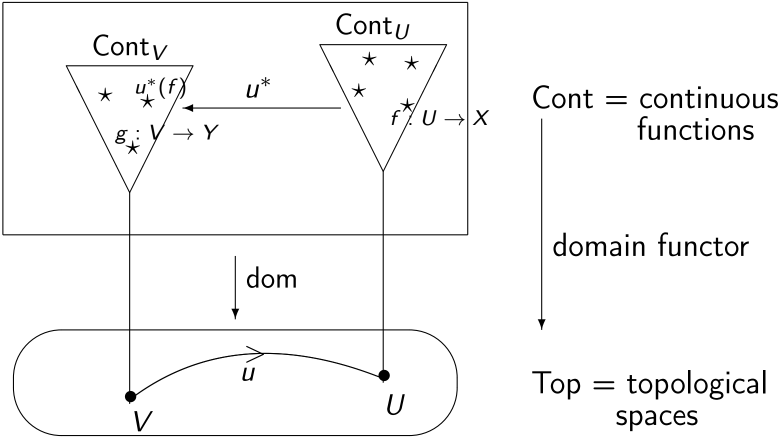
Fibration = category-theoretic generalization of the notion of surjective map.

A *fibration* over a category $C$ is a functor $p : F \to C$ such that

1. For each $C$, $p^{-1}(X)$ is a category written $F_X$ , and called the "fiber above $X$."
2. Each arrow $u : X' \to X$ in $C$ gives rise to a "base change" or "reindexing" functor $u^* : F_X \to F_{X'}$ between the corresponding fibers (in the reverse direction).




A fibration in general




For $f : U \to X$, $\mathbf{u}^*(\mathbf{f}) = \mathbf{f} \circ \mathbf{u} : V \to X$

EXAMPLE

## The codomain fibration

Given a category $C$, let $C^{\to}$ be the category of all arrows in $C$.

### Definition

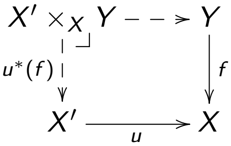
The codomain functor $\mathrm{cod} : C^{\to} \to C$ sends each arrow $f : X \to Y$ in $C$ (= each object of $C^{\to}$) to its codomain $Y$.

### Fact

*The codomain functor is a fibration over $C$, called the "self-indexing" of $C$.*

The fiber above each $X \in \mathrm{Ob}\ C$ contains all the arrows in $C$ with codomain $X$.

The reindexing functors are given by pullback: For each arrow $u : X' \to X$ in $C$, $u^*$ maps any $f : Y \to X$ in $C/X$ to the pullback $u^*(f)$ of $f$ along $u$.



### Definition

Given a fibration $F$ over $C$, a *subfibration of* $F$ is a fibration $G$ over $C$ such that each fiber $G_X$ is a subcategory of $F_X$ and the reindexing functors w.r.t. $G$ are the restrictions of the reindexing functors w.r.t. $F$.

### Fact

*The first axioms of AST ensure that the class $S$ of small maps is a sub-fibration $S \to C$ of the codomain fibration $cod : C^{\to} \to C$.*

# Descent theory

Descent theory = abstract framework geared to describe glueing processes, i.e. the shift from local data to a global item.

Typical example:
Let $(U_i)_{i \in I}$ be a covering of a space $U$, and suppose that for each $i \in I$ a continuous function $f_i : U_i \to V$ is given, in such a way that

$$\forall i, j \in I \quad f_i | U_{ij} = f_j | U_{ij},$$

where $U_{ij} := U_i \cap U_j$.

Then there exists a unique function $f : U \to V$ such that $\forall i \in I \quad f | U_i = f_i$.

# Reformulation in the context of cod : *Cont* $\to$ *Top*

Let's write $U' := \coprod_{i \in I} U_i$, with a canonical map $u : U' \to U$.
The family $(f_i)_{i \in I}$ exactly constitutes an object in $\mathrm{Cont}_{U'}$.

In the following pullback:

$$\begin{array}{ccc} U' \times_U U' & \overset{p_2}{\dashrightarrow} & U' \\ {\scriptstyle p_1} \downarrow & & \downarrow {\scriptstyle u} \\ U' & \underset{u}{\longrightarrow} & U \end{array}$$

- $U' \times_U U'$ represents $\coprod_{i,j \in I} U_{ij}$.
- For $x \in U' \times_U U'$, say $x \in U_{ij}$, $p_1(x)$ is $x$ as a member of $U_i$ and $p_2(x)$ is $x$ as a member of $U_j$.
- For $(f_i)_{i \in I} \in Cont_{U'}$, $p_1^*((f_i)_i) = (f_i|U_{ij})_{i,j}$ and $p_2^*((f_i)_i) = (f_i|U_{ji})_{i,j} = (f_j|U_{ij})_{i,j}$.

So the hypothesis $\forall i, j \in I \;\; \phi_{ij} : f_i|U_{ij} = f_j|U_{ij}$ becomes:

$$p_1^*((f_i)_i) = p_2^*((f_i)_i) \;\text{ in } Cont_{U' \times_U U'} \quad .$$

More generally, replacing identity with an isomorphism, one requires an isomorphism

$$\phi : p_1^*((f_i)_i) \simeq p_2^*((f_i)_i) \text{ in } \mathrm{Cont}_{U' \times U'},$$

i.e., a family

$$\phi_{ij} : p_1^*(f_i) \simeq p_2^*(f_i)$$

of isomorphisms in the fibers above the $U_{ij}$'s (those isomorphisms being compatible when they overlap).

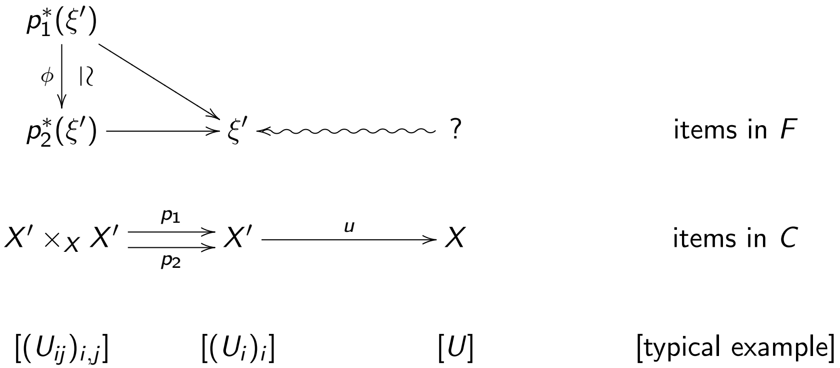
One can glue the local data above the components $U_i$ of the covering $U'$, so as to get a global item above $U = \bigcup_i U_i$.

## Grothendieck, "Technique de descente et théorèmes d'existence en géométrie algébrique" (1959)

Let $F$ be a fibration over $C$.
Given an arrow $u : X' \to X$ in $C$ and $\xi' \in F_{X'}$, a descent datum on $\xi' \in F_{X'}$ w.r.t. $u$ is an isomorphism

$$p_1^*(\xi') \simeq p_2^*(\xi') \text{ in } F_{X' \times_X X'}$$

satisfying compatibility conditions ($\phi_{ik} = \phi_{jk} \circ \phi_{ij}$) expressing that these isomorphisms are compatible when they overlap.




Intuitively, $\xi'$ represents a collection of objects, each one above one of the components of a covering of $X$, and the isomorphism $\phi$ witnesses the agreement of all these objects over all the overlappings of the components of the covering. The question is, then: Can we think of $\xi'$ as the collection of all the restrictions (to all the different components of the covering of $X$) of an actual *single* object above $X$?

If the answer is positive, that is, if $u$ is glueing-friendly, then $u$ is said to be a "descent morphism."

## Definition

Given a fibration $F$ over a category $C$ and an arrow $u : X' \to X$ in $C$, the category of descent data above $X'$ w.r.t. $u$ is the category $\mathrm{Des}_u(X')$:

- whose objects are all pairs $(\xi', \Phi_{\xi'})$, where $\xi'$ is an object of $F_{X'}$ and $\Phi_{\xi'} : p_1^*(\xi') \simeq p_2^*(\xi')$ is an isomorphism in $F_{X' \times_X X'}$;
- whose arrows are all arrows $g : \xi' \to \eta'$ in $F_{X'}$ such that the following diagram commutes:

$$\begin{array}{ccc} p_1^*(\xi') & \xrightarrow{\Phi_{\xi'}} & p_2^*(\xi') \\ {\scriptstyle p_1^*(g)}\downarrow & & \downarrow{\scriptstyle p_2^*(g)} \\ p_1^*(\eta') & \xrightarrow[\Phi_{\eta'}]{} & p_2^*(\eta') \end{array} .$$

For each $\xi \in F_S$, $u^*(\xi)$ is an object of $\mathrm{Des}_u(X')$. The central question of descent theory pertains to the converse: Does any object above $X'$, equipped with a descent datum, can be seen as coming from an object above $X$?

## Definition

The arrow $u$ is called an $F$-descent morphism iff $\rho_u$ is a fully faithful functor. It is called a strict $F$-descent morphism (a.k.a. an *effective $F$-descent morphism*) iff $\rho_u$ is an equivalence of categories.

In other words, an arrow $u$ in $C$ is a strict $F$-descent morphism iff giving an object in $F_X$ exactly amounts to giving an object in $F_{X'}$ endowed with a descent datum.

What virtually glues along $u$ actually glues.

# Link with AST

### Definition

An arrow $u : X' \to X$ is a strict descent morphism if it is a strict $F$-descent morphism for $F = \ C^{\to} \xrightarrow{\text{cod}} C$ .

### Proposition (Grothendieck 1959, Proposition 2.1)

*If the category $C$ has finite products and pullbacks, then the strict descent morphisms in $C$ are exactly the (universal strict) epimorphisms in $C$.*

Reformulation of Joyal-Moerdijk's Axiome 3:
Any epimorphism in $C$ is a strict descent morphism for the sub-fibration $p_S : S \to C$ of the codomain fibration $\text{cod} : C^{\to} \to C$.

**Theorem**

*Let $C$ be a Heyting pretopos. A collection of arrows in $C$ satisfies the first three axioms of AST iff $S$ defines a subfibration of $cod : C^{\rightarrow} \rightarrow C$ and the strict $S$-descent morphisms in $C$ are exactly the epimorphisms in $C$.*

In other words, a class of small maps of $C$ is partially characterized as a self-sub-indexing of $C$ for which the analog of Grothendieck's result holds.

## Reinterpretation of AST's first three axioms:

- Axiom 1 says that all isomorphisms in $C$ belong to $S$, which implies that, for any object $A$ of $C$, $1_A$ belongs to $S$, and thus that there is at least one object (arrow) in $S$ above each object of $C$. So the category $S$ can really be said to be "above" $C$.
- Axiom 2 then says that any pullback of any arrow in $S$ belongs itself to $S$, which ensures that $S$ induces a sub-fibration of $\mathrm{cod} : C^{\to} \to C$.
- Axiom 3 finally says that Grothendieck's result about $\mathrm{cod} : C^{\to} \to C$ remains true about $\mathrm{cod}|_S : S \to C$.

A class of small maps can be partially characterized as a category to which descent theory applies.

# Stacks

In the context of descent theory, the natural extension of a fibration is a stack.
If a fibration is a family of categorical objects which “pullback like bundles,” a stack is a family of such objects which moreover “glue like bundles.”
The notion of stack, however, makes sense only on the condition that the base category $C$ is endowed with a Grothendieck topology.

## Definition

Given a category $C$, a Grothendieck topology $K$ on a category $C$ consists of a collection $K(X)$ of “covering families” $\{f_i : X_i \to X\}_{i \in I}$ on $X$, one collection for each object $X$ of $C$, in such a way that the different $K(X)$'s are connected in the coherent way.
A site is a category endowed with a Grothendieck topology.

### Definition

Given a site $\langle C, K \rangle$, a stack over $C$ is basically an indexed category over $C$ such that any covering family $\{f_i : X_i \to X\}_{i \in I}$ in $K(X)$ amounts to a strict descent morphism.

Important examples of stacks are:

- the fibration *Cont* over *Top*, when *Top* is endowed with the Grothendieck topology defined by the open covers (in the usual sense);
- the codomain fibration $C^{\to}$ over $C$, when $C$ is endowed with the "coherent topology" $K_c$, i.e., the Grothendieck topology defined by all the finite jointly epimorphic families of arrows.

### Theorem

*Let $C$ be a Heyting pretopos $C$ and $S$ a class of arrows in $C$.*

*Then $S$ satisfies the first three axioms of AST iff $S$ is a stack over the site $\langle C, K_e \rangle$,*
*where $K_e$ is the Grothendieck topology defined by covering families each of which consists of a single epimorphism.*

In what follows, I will sketch some extensions of the connection established by AST between logic and algebraic geometry.

# Extension 1: Revisiting the notion of a universal set

The import of descent theory by AST comes with the advantage of addressing one important issue faced by set theory, namely the need to formalize unbounded, class-like quantification.

The original motivation behind AST is the remark that, in the internal language of any topos, unbounded quantification is by construction excluded: Only quantification restricted to an object is available.

In that perspective, the distinction between set-like objects and class-like objects should be formalized in any topos, so that class-wide quantification can be emulated.

On that score, a further axiom of AST, the "Representability Axiom," introduces a "locally universal" object $U$, so that it becomes possible to interpret unbounded quantifiers ranging over all small objects by using bounded quantifiers ranging over $U$.

The Representability Axiom is:

There exists a map $\pi : E \to U$ in $S$ such that, for any map $f : Y \to X$ in $S$, there exists a diagram

$$\begin{array}{ccccc} Y & \longleftarrow & Y' & \longrightarrow & E \\ {\scriptstyle f}\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow{\scriptstyle \pi} \\ X & \underset{p}{\twoheadleftarrow} & X' & \longrightarrow & U \end{array}$$

in which $p$ is an epimorphism and both squares are pullbacks.

## Definition

Given a class $\mathcal{K}$ of arrows in a pretopos $C$, a $\mathcal{K}$-classifier $\pi_{\mathcal{K}}$ is an arrow $\pi_{\mathcal{K}} : E_{\mathcal{K}} \to U_{\mathcal{K}}$ such that, for any arrow $f : Y \to X$ in $\mathcal{K}$, there exists an essentially unique arrow $c_f$ fitting into the following pullback:

$$
\begin{array}{ccc}
Y & \longrightarrow & E_{\mathcal{K}} \\
{\scriptstyle f}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathcal{K}}} \\
X & \xrightarrow[c_f]{} & U_{\mathcal{K}}
\end{array} \quad .
$$

The $\mathcal{K}$-classifier is said to classify, for each object $X$ of $C$, the arrows in $\mathcal{K}$ above $X$. It is also called a universal family w.r.t. $\mathcal{K}$. If only the existence of an arrow $c_f$ is required, the arrow $\pi_{\mathcal{K}}$ is called a versal family w.r.t. $\mathcal{K}$.

A subobject classifier for $C$ is nothing but an $\mathcal{M}$-classifier, wherein $\mathcal{M}$ is the class of all monomorphisms in $C$. But there are many other cases.

An object classifier for $C$ is a $C^{\rightarrow}$-classifier, i.e., a classifier w.r.t. the class of *all* arrows in $C$.

For each object $X$ of $C$, such an object classifier, should it exist, would classify all the objects over $X$, i.e., all the maps $f : Y \rightarrow X$ with codomain $X$.

In particular, every object of $C$ would be a pullback of $U_{C^{\rightarrow}}$, thus revealed as a "universal object."

For reasons related to Russell's paradox, the existence of an object classifier in a topos would lead to a contradiction.

Given a class $S$ of small maps in $C$, a global $S$-classifier cannot be a small map itself. The reason is just a variant of the argument above.

But the Representability Axiom only posits the existence, among all small maps, of a local $S$-classifier.
(More specifically, the posited small map $\pi : E \to U$ is a locally versal family w.r.t. $S$, and accordingly $U$ is a locally versal small object, which "locally contains all small objects.")
The resulting theory still is equiconsistent with ZFC.

In view of this, localization appears as a powerful way to reconsider the issue of a universal set, and to get an optimal solution, namely a locally versal set.

This solution relies on the conception of sets and maps as the results of constructions from local data, and thus directly on the fact that AST builds descent theory into set theory.

## Extension 2: Revisiting identification in mathematics

Mathematical identification takes on a well-know figure: the identification of different items by shifting to the quotient of an equivalence relation. This is identification by abstraction.

But descent theory brings out another figure of identification in mathematics: the construction of a single global entity from different localizations of it, viewed as being various partial perspectives on the same thing. This is identification by glueing.

In descent theory, one manipulates a whole descent datum

$$\langle \xi' \in F_{X'}, \Phi_{\xi'} : p_1^*(\xi') \simeq p_2^*(\xi') \rangle$$

which corresponds to the system of all the components of a covering of some virtual identical object $\xi \in F_X$. No single component is considered in isolation.

On gets a presumed object from local data, which are both anticipated and then recaptured as being so many different aspects of the same thing.

Carlos Simpson, "Descent":

> *Modern geometry springs from the observation that different map views have to be "glued together" on the overlapping territory, to obtain a global picture. [. . . ] The different pieces making up a space don't have to share a common existence in a single place. The glueing data, abstracted into the categorical structure of the topos of sheaves, provide the links which form a virtual reality from which the geometric object emerges. The original "ground level" fades from view, replaced by the abstract collection of glueing data itself as the only true reality.*

Carlos Simpson (cont'd):

> *When we think of the complicated "ground level" as being a reality that is best apprehended (and might only really exist) through a covering representing a wide variety of different approaches and viewpoints, the place where things are really happening, and what we should concentrate on understanding, is the glueing data which explain how to pass between the various different points of view and how they are bound together.*

The idea is that local data are naturally to be understood as perspectives on some single global entity which is nothing beyond its reconstruction through the glueing process of its different aspects.

Descent theory provides a figure of identification which should be recognized as irreducible to any other.

# Extension 3: Revisiting coreferentiality in language

How to understand that an identity such as "Cicero is Tully" may be an informative statement?

1. If the sentences "Cicero = Cicero" and "Cicero = Tully" are semantically different, the names "Cicero" and "Tully" must be semantically different. (Compositionality)
2. If the names "Cicero" and "Tully" are semantically different, they are referentially different.
3. The names "Cicero" and "Tully" are not referentially different, but coreferential.

The Fregean will reject assumption 2: The two names are coreferential, and yet are semantically different, because they have two different senses.

The Referentialist will reject the assumption that the two sentences are semantically different.

# Kit Fine, *Semantic Relationism* (2007)

Kit Fine endeavors to defend a third option, a more satisfying one, by rejecting the assumption 1 (Compositionality).
His idea, briefly, is this:

- There is an intrinsic semantic difference between the sentences "Cicero = Cicero" and "Cicero = Tully", because there is one between the pairs "Cicero", "Cicero" and "Cicero", "Tully".
- This difference is based on the fact that the first pair represents the same object as the same whereas the second pair does not.
- There is only an extrinsic semantic difference between the names "Cicero" and "Tully".
  The two names differ only in respect of whether their pairing with "Cicero" gives something with the same semantic status as "Cicero", "Cicero". There are relational aspects to the meaning of names.

# Coordination

Fine calls *coordination* the fact of representing as the same.

The sentences "Cicero = Cicero" and "Cicero = Tully" both represent the objects "as being the same", but only the first represents them "as the same."

Representing as the same is the result of a semantic requirement, owing to which two expressions are coordinated.

A *coordination scheme* consists of an equivalence relation on the occurrences of names in a sequence of propositions, by which it is required that the related names should corefer.

Fine seeks to develop a semantics that would be able to express semantic relationships on top of intrinsic semantic values. But his proposal is but very tentative.

In a way, Fine does not so much give the solution as simply name it.

But one cannot help but think of descent theory as a toolkit to flesh out the semantical framework suggested by Fine.

It amounts to comparing the lines of coordination that make up a coordination scheme with the glueing isomorphisms that make up a descent datum.

## Conclusion

- AST reinterprets set theory so as to turn it into an arrow-based theory, along the codomain fibration.
- AST carries out the graft of descent theory onto ZFC.
- This graft lends itself to several extensions:
  1. It casts the issue of an universal set in new terms.
  2. It puts forward an original figure of identification in mathematics.
  3. It holds out the prospect of new tools for the semantics of language.