Fibered structures for modal logic

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Topic of today's talk: modalities

Modalities = possibility and necessity (as well as impossibility and contingency).

Modal notions are philosophical notions par excellence.
They are not mathematical ones per se. (I leave aside modalized toposes –Lawvere-Tierney topologies– or provability logics.)

Modalities have several possible meanings (logical, metaphysical, epistemic, . . .), but I will endorse a neutral approach in this respect.

I will simply take them to be propositional features (features applied to truths and falsities): Modalities describe how something is true.

Modal iteration

It is the superposition of modal clauses, as when some proposition is said to be necessarily necessarily true, or (other example) necessarily possibly necessarily true.

One very quickly feels dizzy about such ideas.

Two possible reactions:

- Those ideas are crazy.
- The philosophical study of modal notions requires formal tools (since Aristotle's *First Analytics*).

I will rather follow the second way, but let me say a word about the first one.

The Leibnizian conception of modalities (possible worlds)

The actual world is just one possible world among others —each of which could have been elected as the actual world, should God have not seen that *this* actual world is the best possible world.

In Leibniz's perspective, the determination of the best possible world is an optimization problem that has a unique solution and supposes that all the possible worlds whatsoever make up an homogeneous and complete set within which the actual world can be singularized.

In particular, Leibniz's theodicy excludes that there could be other collections of possible worlds than the one that happens to exist. Otherwise, God could be reproached with having not chosen a collection of possible worlds, the best of which would be better than the actual world.

The Leibnizian conception of modalities (cont'd)

Important postulate: Possible worlds make up a single-level collection.

In Leibniz, the single-levelness of all possible worlds goes hand in hand with their absoluteness (= the actual collection of possible worlds is the only collection there is).

To sum up:

- necessity, possibility and contingency are absolute propositional features;
- possible worlds all lie at the same level, namely God's infinite understanding;
- iterated modalities are meaningless.

Logical necessity

Wittgenstein, *Tractatus*, 5.525:

> *Certainty, possibility or impossibility of a state of affairs are not expressed by a proposition but by the fact that an expression is a tautology, a significant proposition or a contradiction.*

There is but one single logical space.

Carnap, *Meaning and necessity*: Modal iteration is not even considered, because logical validity cannot be relativized.

Robert Stalnaker, “Merely Possible Possible Worlds” (2012)

Stalnaker conceives of possible worlds as maximal (consistent sets of) propositions. Since singular propositions about contingently existing individuals are themselves contingent, possible worlds are contingent objects.

But

> *if we accept the thesis, we need to consider its effect on a semantics for modal notions. [. . .] actualism is threatened by the problem of iterated modalities.*

Conclusion: Iterated modalities do not go without saying, philosophically speaking.

The parlance of "possible worlds"

Thinking of modalities in terms of possible worlds (at least heuristically) has become customary. In that parlance:

ϕ could have been possible
$= \phi$ is not possible but, were the whole actual constitution of the possible replaced by another one, ϕ would become possible.

In terms of possible worlds semantics: The datum which is the collection of all possible worlds should be itself counterfactualizable
= The actual collection of all possible worlds is only one among other possible collections of possible worlds.

This calls for a concept of HIGHER ORDER possible world.

(Modalities pertain to truths as being not only factual, and yet the collection of all possible worlds is referred to by Leibniz as a kind of super-fact.)

Summary

Iterated modalities remain a philosophical object, even if (all the more) they are rejected for philosophical reasons.

The concept of higher order possible world can be understood as a form of *reductio*. Philosophers averse to iterated modalities basically say:

> *"Iterated modalities make sense only if we can conceive of some kind of higher order possible world. But this concept is stricken with too many difficulties. So iterated modalities should be rejected."*

In what follows, I simply will postulate that modal iteration makes sense, in order to formalize (semantically) the concept of higher order possible world and to show that it is semantically much more demanding than modern modal semantics makes it out to be.

The predicament of modern modal logic

Contrary to Leibniz, it allows for modal iteration. Syntactically, modalities become operators, so their iteration becomes their raison d'être (despite the aim of formalizing the expression of modalities in ordinary language).

$$\Diamond p = \text{“}p \text{ is possibly true”}$$

$$\Box p = \text{“}p \text{ is necessarily true”}$$

$$p, \Diamond p, \Diamond\Diamond p, \Diamond\Diamond\Diamond p, \ldots$$

$$p, \Box p, \Box\Box p, \ldots, \Box\Diamond\Box p, \ldots$$

On the other hand, modal logic retains a crucial component of Leibniz's conception (despite the departure from metaphysics). Semantically, modal logic assumes that all possible worlds make up a *single-level* collection of all possible worlds and thus that higher order possible worlds make no sense.

Modal axioms

A system of modal logic is normal if it includes the following as fundamental axioms:

- all the tautologies of nonmodal propositional logic
- axiom **K**: $\Box(p \to q) \to (\Box p \to \Box q)$
- axiom **T**: $\Box p \to p$

Inference rules: modus ponens, the necessitation rule (if ϕ is a theorem, so is $\Box\phi$) and the rule of uniform substitution (if a proposition $\chi(p)$ containing p is a theorem, so is $\chi[\phi/p]$, for any proposition ϕ).

The ensuing system is called T.

The addition of the axiom **4**, $\Box p \to \Box\Box p$, defines the system S4.

The further addition of the axiom **5**, $\Diamond p \to \Box\Diamond p$, gives the system S5.

Modal axioms tame modal iteration.

Kripke semantics (1963)

Kripke semantics, based on the use of an "accessibility relation," can be seen as the optimal way to allow for modal iteration while maintaining the single-levelness of all possible worlds.

- W = fixed set of "possible worlds"
- R = "accessibility relation" defined on W
  $wRw' = w'$ is an admissible variant of w
- For each possible world w, one specifies which (atomic) propositions are true in w ("valuation" on W).
- $\Diamond p$ is true in w iff p is true in some possible world w' accessible from w (wRw').
- $\Diamond\Diamond p$ is true in w iff p is true in some world w'' accessible from some world w' accessible from w ($wRw'Rw''$).

In Kripke semantics, modal iteration is linearized. It translates into a ramification: the higher the modal degree of the proposition, the longer the branches of the graph of R to consider.

The Single-Levelness Postulate

A correct understanding of modal iteration requires abandoning such a postulate, because it urges to consider different levels of possibility.

Indeed, saying that a proposition is necessarily necessarily true amounts intuitively to saying that the proposition is necessarily true whatever the range of the possible may be.

In the same way, when one says about a state of affairs that it is not, but could have been possible, one implicitly shifts from a given system of possibility into a context in which that system is relocated as being only one among others, and one claims that with respect to some other system, the state of affairs becomes possible.

In terms of possible worlds:

- truth = satisfaction in the actual world w_0.
- necessary truth = satisfaction, not only in w_0, but in any possible world in W (for simplicity's sake, every world is supposed here to be accessible from the actual one).
  Shift from w_0 to W = change of scale.
- necessarily necessary truth = ?
  Answer: It should correspond to an analogous change of scale, pointing to a set W^2 of sets of worlds, which would be to W as W is to w_0.
- And so on, in case of any further modal iteration.

(See Kit Fine's formalization of higher order vagueness.)

Claims:

- Modal iteration can be refused for very legitimate reasons. But as soon as it is accepted, it should be recognized a very strong meaning: It prompts a semantic change of scale in the sense of shifting from one whole configuration of the possible to another.
- Kripke semantics does not match that meaning: It is either too much (unlimited iterability) or too little (does not account for modal change of scale).
  Kripke semantics remains too Leibnizian (keeps the single-levelness postulate) or not enough (accepts modal iteration).

Goal:

Putting forward a new possible worlds semantics for propositional modal logic that would account for modal change of scale (against the single-levelness postulate).

This new semantical framework calls for new mathematical structures.

Wanted:

An open-ended collection of nested systems of possibility, where each system corresponds to a set of possible worlds lying at some level.

Applications of fibered structures in logic

Working hypothesis 1: Logic has always been acting as the interface which makes it possible for philosophy and mathematics to meet. This mediating role of logic is not altered by its division into "philosophical logic" and "mathematical logic."

Working hypothesis 2: The technical tools used by philosophical logic has been artificially sealed and does not interact with mathematics in a uniform way.

Logic would benefit from importing mathematical structures hitherto underutilized, especially fibered structures (fiber bundles, fibered categories).

Fibered structures are geared to expressing context change (base change functors), which is often what logic is about.

Problem:

How could a mere set of possible worlds could make up a higher order possible world (i.e. something structured and unified as a world)? How to make up a world with a multiplicity of worlds?

Solution: We should turn things upside down.

Let's rather think of each possible world as being virtually a set of possible worlds of higher-level (instead of thinking of a world of higher-level as being a set of worlds of lower-level).

Guiding picture:

- Higher-level possible worlds are specifications of lower-level worlds.
- Each lower-level world is the set of all its possible specifications at the next level. (Think of all possible Taylor approximations of functions having some given value at 0.)

Analysis, not synthesis: A world becomes a multiplicity of worlds insofar as the analysis of a certain structure at that world uncovers such a multiplicity.

A higher order possible world is a basic (level 0) world, viewed as the index of some space containing higher-level possible worlds. (Distinction level/order.)

Wanted 2: a semantic framework

- comprising an open-ended collection of possible worlds of increasing levels
- where, at each modal iteration, a coherent set of possible worlds (system of possibility) is considered relatively to some already given possible world
- so that the interpretation of a proposition of modal degree k involves nested worlds of respective levels $0, 1, \ldots, k$ and of respective orders $k, k-1, \ldots, 0$.

(In Kripke a possible world is not intrinsically relative to some system of possibility: It can lie at the end of accessibility lines of different lengths, so that levels of possibility cannot really be distinguished.)

Candidate: a geometric manifold.

Many cues for differential geometry:

- To each point x of M is attached the tangent space to M at x, written T_xM. T_xM is composed of vectors.
- TM, the disjoint union of all the tangent spaces, is itself a manifold.
- Iterability: Hence $T^2M := TTM, T^3M, \dots$ can be introduced.
- Change of scale: TM is twice as dimensional as M.
- The order of a world becomes comparable to the order of a Taylor expansion.

Basic scheme:

- 0-level possible worlds = points x of a given manifold M
- 1-level worlds relative to $x \in M$ = members of $T_x M$ (then x is considered as being of order 1)
- TM = set of all 1-level worlds in general
- accessible worlds = path-connected points (for a certain selection of paths)
- interpretation of an atomic proposition p = set of curves on M
- p is true at x iff one of these curves passes through x
- interpretation of a proposition ϕ of modal degree k = set of curves on $T^k M$
- ϕ is true at x iff one of these curves passes, not through x (impossible), but through some point of a set of points $\pi^k(x) \subseteq T^k M$ connected to x.

TASKS:

1. Interpreting of $\neg\phi$, $(\phi \wedge \psi)$ and $(\phi \vee \psi)$.
2. Defining modal lift: curves on M interpreting ϕ lifted into curves on TM interpreting $\Box\phi$.
3. Defining nonmodal lift. Example of $(\Box p \wedge q)$: the curves interpreting q have to be lifted onto TM to become homogeneous to the curves interpreting $\Box p$. Yet $\Box q$ does not occur in the proposition.
4. Associating, to each point $x \in M$, a sequence $\pi^0(x) = \{x\}, \pi^1(x) \subseteq TM, \ldots, \pi^k(x) \subseteq T^kM, \ldots$ of counterparts of x
5. Checking validities.

Need of some further structure on M!

Technical definitions

Riemannian manifold = manifold M endowed with a *metrics*, that makes it possible to measure the distance between any two points of M, and thus to define which are the geodesics (= the "straight lines" on M).

Any metrics on M gives rise to a natural connection ∇, which allows one to connect vectors from some tangent space to vectors from some other tangent space.

Intuitively, ∇ is the operator that assigns, to any couple X, Y of vector (fields), the infinitesimal deviation of Y w.r.t. the direction indicated by X.

Technical definitions, cont'd

A vector field X defined along a curve γ is said to be parallel if $\nabla_X X = 0$ at any point of γ.
That means that X never deviates from itself.

There is a natural projection $p : TM \to M$ which sends any tangent vector $v \in T_x M$ to the point x of M at which it is tangent.

Fact: For any geodesic $\gamma : \mathbb{R} \to M$ on M and any tangent vector v at $\gamma(0)$, there exists (locally) a unique curve $\delta : \mathbb{R} \to TM$ on TM such that:

- δ lies above γ (i.e. $p \circ \delta = \gamma$)
- δ passes through v (i.e. $\delta(0) = v$)
- δ (viewed as a vector field along γ) is parallel.

The curve δ is called the horizontal lift of γ through v at $t = 0$, written $\widetilde{\gamma}^v$.

Technical definitions, cont'd

Fact: Any metrics g on M gives rise to a natural metrics g_T on TM.

Iteration: Starting from a Riemannian manifold $\langle M, g \rangle$,

- $M_0 = M$ and $g_0 = g$
- $M_{n+1} = T(M_n)$ and $g_{n+1} = (g_n)_T$
- $p_{n+1} : \langle M_{n+1}, g_{n+1} \rangle \to \langle M_n, g_n \rangle$ is the natural projection.

Technical definitions, the end

Last fact: The connection ∇ allows one to define the parallel transport along a curve γ of any vector v in $T_{\gamma(t)}M$ into a vector $J^{\gamma}_{t,t'}(v)$ in $T_{\gamma(t')}M$.

The parallel transport connects possible worlds of the same level but lying above different points (= belonging to different systems of possibility).

TASK 4: Defining a sequence of counterparts of x above each $x \in M$.

To that end, $\langle M, g \rangle$ is endowed with a family $\mathcal{A} = \{\gamma_i : i \in I\}$ of accessibility curves on M.

Set of accessible worlds from x:

$$A^1(x) := \bigcup_{x \in \overline{\gamma_i}} \overline{\gamma_i}$$

($\overline{\gamma} := \{\gamma(t) : t \in \mathbb{R}\}$ = set of all the points of the curve γ. Likewise, for any set of curves Γ, $\overline{\Gamma} := \bigcup_{\gamma \in \Gamma} \overline{\gamma}$.)

TASK 4 fulfilled

Idea: The counterparts of x in TM come from the infinitesimal data at x sending x to a world accessible from x.

There is a way of coding the points of M accessible from x with elements of TM. For $v \in T_xM$, let c_v be the geodesic of M starting at x with speed v.

$P^1(x) := \{v \in B_x : c_v(1) \in A^1(x)\}$ = set of proxies of x in T_xM.

$\pi^1(x) := \{c_v'(t) : v \in P^1(x), t \in \mathbb{R}\}$ = set of all 1-counterparts of x.

(The map $(v, t) \mapsto c_v'(t)$ is called the "geodesic flow" of $\langle M, g\rangle$.)

The construction of π^1 can be iterated:

$$\pi^2 : TM \to \{\text{curves of } TTM\}$$

Indeed, it suffices to replace $\langle M, g \rangle$ with $\langle TM, g_T \rangle$, and to replace the accessibility curves γ_i on M with 2-accessibility curves $J_w^{\gamma_i}$ on TM.

The connection attached to the Riemannian manifold M is what supports the lift of accessibility curves on M into accessibility curves on TM.

The elements of $\pi^n(x)$ will be called the *n*-counterparts of x.

TASKS 2 & 3: Defining the modal and nonmodal lifts of any curve on M.

To that end, $\langle M, g \rangle$ is endowed, for each $n \geq 0$, with a selection $\mathcal{C}_n$ of admissibles curves on $T^n M$.

The members of $\mathcal{C}_n$ are all the curves from which the interpretation of any proposition of modal degree n is bound down to being drawn.

Natural conditions on the $\mathcal{C}_n$'s:

- for any $v \in T^n M$ there exists at least one $\gamma \in \mathcal{C}_n$ passing through v
- $\delta \in \mathcal{C}_{n+1}$ implies $p_{n+1}(\delta) \in \mathcal{C}_n$
- for any $\gamma \in \mathcal{C}_n$, there exists at least one $\delta \in \mathcal{C}_{n+1}$ lifting γ (i.e. $p_{n+1}(\delta) = \gamma$).

Any curve γ on $T^n M$ gives rise to two families of curves on $T^{n+1}M$.

The modal lift of a curve is the family of all its horizontal lifts:

$$\lambda(\gamma) = \{\widetilde{\gamma}^{k\gamma'(t)} \in \mathcal{C}_{n+1} : \gamma \in \Gamma, k \in \mathbb{R}^*, t \in \mathbb{R}\}$$

The nonmodal lift of a curve is its derivative:

$$L_n^{n+1}(\gamma) = \{\gamma'\} \cap \mathcal{C}_{n+1}$$

Since the gap in modal degree between two principal subformulas of a single proposition can be > 1, iterated nonmodal lift has to be defined:

$$L_n^n = \mathrm{id}, L_n^{n+m} = L_{n+m-1}^{n+m} \circ \ldots \circ L_n^{n+1}.$$

A metric modal frame $\underline{F}$ is a quadruple $\langle M, g, \mathcal{A}, (\mathcal{C}_n)_{n\geq 0}\rangle$.

A valuation V on $\underline{F}$ is the assignment, to any atomic proposition p, of a subset $V(p)$ of $\mathcal{C}_0$.

A metric modal model is a metric modal frame endowed with a valuation.

TASK 1: Inductive definition of $V(\phi)$

- $V(\neg\phi) = \mathcal{C}_n \setminus V(\phi)$ (where $\deg(\phi) = n$)
- $V(\phi \wedge \psi) = V(\phi) \cap L^n_m(V(\psi))$ (where $\deg(\phi) = n$ and $\deg(\psi) = m < n$)
- $V(\phi \vee \psi) = V(\phi) \cup L^n_m(V(\psi))$ (under the same hypothesis as above)
- $V(\Diamond\phi) = \lambda(V(\phi))$

Definition

For any proposition ϕ of modal degree n and for any $x \in M$:

$$\langle \underline{E}, V \rangle, x \vDash \phi \quad \text{iff} \quad \pi^n(x) \cap \overline{V(\phi)} \neq \emptyset.$$

In other words, a modal proposition ϕ is true at x if one of the possible worlds relative to x that are prompted by the construction of $V(\phi)$ coincides with some counterpart of x at the level defined by the modal degree of ϕ.

Simple case

Let $\mathbb{P}$ be the Euclidean plane endowed with a referential, with all the horizontal lines as the accessibility curves.

Let $O = (0, 0)$ be the point taken as the origin of $\mathbb{E}^2$.

One gets:

$\mathbb{P}, O \vDash \Diamond p$ iff there exists a curve γ_p in $V(p)$ that crosses the x-axis at least once (even if it is not at O).

This clause is reminiscent of Kripke semantics.

Kripke semantics recaptured

Each Kripke frame $\mathcal{K} = \langle W, R \rangle$ induces the metric frame $\underline{F}^{\mathcal{K}} = \langle M^{\mathcal{K}}, g^{\mathcal{K}}, \mathcal{A}^{\mathcal{K}}, (\mathcal{C}_n^{\mathcal{K}})_{n \geq 0} \rangle$ defined by:

- $M^{\mathcal{K}}$ is obtained by taking the graph of R and then its "straight-line drawing" in some Euclidean space $\mathbb{R}^d$.
- $g^{\mathcal{K}}$ is the Euclidean metrics induced by $\mathbb{R}^d$.
- $\mathcal{A}^{\mathcal{K}}$ is the set of all vertices and all straight-line segments in the straight-line drawings of the graph of R.
- $\mathcal{C}_0^{\mathcal{K}}$ is $\mathcal{A}^{\mathcal{K}}$ and $\mathcal{C}_n^{\mathcal{K}}$ is the set of all straight-line segments L of $T^n\mathbb{R}^d$ such that $(p_1 \circ p_2 \circ \ldots \circ p_n)(L) \in \mathcal{A}^{\mathcal{K}}$.

Kripke semantics amounts to disregarding the space that underpins the web of accessibility relations between the possible worlds.

Theorem

For any Kripke frame $\mathcal{K}$ and any modal proposition ϕ,

$$\underline{F}^{\mathcal{K}} \models_{GML} \phi \ \ \textit{iff} \ \ \mathcal{K} \models_{Kripke} \phi.$$

The modal semantics based on metric models thus generalizes Kripke semantics.

Three kinds of modal semantics

The number of matches between an evaluation world x and a valuation $V(\Diamond\phi)$ is supposed to be generally greater than the number of matches between x and $V(\phi)$. There are three way to achieve this:

- $V(\Diamond\phi)$ enlarges $V(\phi)$, while the evaluation world x is left unchanged.
  This is what happens in Tarski-McKinsey semantics, where $V(\Diamond\phi) =$ topological closure of $V(\phi)$.
- $V(\phi)$ is not modified, but the shift to $\Diamond\phi$ results in x being enlarged into a whole set of worlds.
  That is what happens in Kripke semantics.
- Both $V(\phi)$ and x are enlarged.
  That is what happens in the present framework: $V(\Diamond\phi)$ enlarges $V(\phi)$, and x is enlarged into $\pi^1(x)$ as well (and again at the higher levels).

TASK 5: Validity and completeness results

Axioms **T** and **K** are valid.

The axiom **4** is not valid in general.

However, the axiom **4** becomes valid if one restricts oneself to a class of very special metric modal frames.

Definition

A metric (modal) frame $\underline{F} = \langle M, g, \mathcal{A}, (\mathcal{C}_n)_{n \geq 0} \rangle$ is sparse if the following conditions hold:

(i) $\langle M, g \rangle$ is a simple Riemannian manifold (i.e. any two points of M are related by at most one geodesic);

(ii) $\mathcal{A}$ and $\mathcal{C}_0$ are composed of geodesics only;

(iii) for any $n \geq 0$, $\mathcal{C}_{n+1}$ is the set of all the horizontal lifts of curves in $\mathcal{C}_n$.

Proposition

The axiom **4** *is valid in all sparse metric frames.*

The rule of uniform substitution does not preserve validity, but that is in order.

Indeed, when one is committed to some modal claim (such as $\Box\Box p \rightarrow \Box p$), one should not be committed to the analogous claim about propositions of higher modal degree than that of p.

Closure under substitution is motivated by the belief that if one is able to establish something about simple necessity, one is by the same token entitled to admit the same thing about necessary necessity.

But the claim that the meaning of necessity is uniform is not self-evident at all.

S4* = S4 with the rule of uniform substitution restricted to the case where the substituens is of modal degree not higher than that of the substituendum.

S4** = S4* without the necessitation rule.

Theorem

*The system S4** is complete w.r.t. the class of all sparse metric frames.*

Definition

An Hadamard metric frame is a metric frame $\underline{F} = \langle M, g, \mathcal{A}, (\mathcal{C}_n)_{n\geq 0}\rangle$ such that:

(i) $\langle M, g\rangle$ is a simply connected, complete Riemannian manifold of nonpositive sectional curvature;

(ii) $\mathcal{A}$ and $\mathcal{C}_0$ are composed of geodesics only;

(iii) for any $n \geq 0$, $\mathcal{C}_{n+1}$ is the set of all the horizontal lifts of curves in $\mathcal{C}_n$.

Theorem

The system S4 is complete w.r.t. the class of all Hadamard metric frames.*

Thus, the partial neutralization of modal iteration expressed by the axiom **4** corresponds to a really very special class of metric frames (more so than in Kripke semantics).

S5* = S5 with the rule of uniform substitution restricted as in the case of S4*.

Definition

A sparse metric frame $\underline{F} = \langle M, g, \mathcal{A}, (\mathcal{C}_n)_{n\geq 0}\rangle$ is minimal if any two geodesics in $\mathcal{A}$ are either disjoint or identical (up to linear reparameterization).

Theorem

The system S5 is complete w.r.t. the class of all minimal metric frames.*

Conclusion

(I) Philosophical thesis:

- A more faithful account of modal iteration does not imply abandoning a possible worlds semantics, but it surely implies abandoning the Leibnizian heritage of the Single-Levelness Postulate.
  Despite the introduction of accessibility relations, Kripke semantics still endorses that heritage.
- A notion of higher order possible world is needed to fully account for modal iteration.
  Higher order possible worlds make up a collection of possible worlds organized along indefinitely many increasing levels.
- Only a fibered framework is capable of providing a real semantic representation of such higher order possible worlds.

Back to Robert Stalnaker

Stalnaker's point is to "think of the [possible] worlds, not as the *points* in logical space but as the cells of a relatively fine-grained partition of logical space [...]. We do not thereby foreclose the possibility that in some other context, one might cut the space up more finely."

Iterated modalities are contexts that require to cut up the space up more finely.

In Stalnaker's setting, two points within the same partition cell "have exactly the same representational significance" but may be distinguished by higher order propositions.

In geometric modal logic, a possible world, as a point of a given manifold, is characterized by all the propositional curves to which it belongs, but may be more finely characterized by the derivative and higher order derivatives of those curves.

Related remark

Two propositions ϕ and ψ, interpreted as sets of curves, may have coextensional valuations (pass through the same 0-worlds) and yet have non coextensional possibilizations $V(\phi)$ and $V(\psi)$

This is because each curve takes into account the way in which it passes through each world that it contains.

Conclusion and the end

(II) The second part of my talk has been devoted to building up a semantic framework in keeping with the philosophical thesis.

- The resulting framework is much more faithful than Kripke semantics to the idea of nested systems of possibility and thus to meaning of modal iteration.
- It is technically complex (more difficult to handle than Kripke semantics). But that is not the point.
- It carries an important geometric twist, replacing Kripke's accessibility with a geometric connection.
- It generalizes in a certain way Kripke semantics.
- It shows interesting similarities between modal operators and differential operators.
- It gives rise to geometric completeness results about variants of S4 and S5.