Fibrational Semantics

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Introduction

The question "What can Set Theory do for Philosophy?" would never be asked nowadays.
The main reason for dismissing such a question is the fact that philosophical logic has become mainly set-theoretic since Tarski.

An implicit diagnosis has been that set theory is for philosophers interested in logic, whereas category theory is for philosophers interested in mathematics.
This of course has proved to be completely false – in particular because logic is historically in charge of the interaction between philosophy and mathematics, and so cannot be used as a firewall against actual mathematics!

Category theory should not get out of the way of set theory when it comes to philosophical logic: it is perfectly compatible with set theory and opens logic to all branches of mathematics.

Category theory can do something for philosophy – provided that it keeps being embedded into philosophical logic.

But why should category theory do something really useful for philosophical logic? Why would category theory be the only way to achieve it?

Contextuality plays an important role in Tarskian logical semantics. Model theory relies on the consideration of various “universes of discourse” and on the consideration of what remains the case from one universe to another.

Now, precisely, category theory, among other things, provides the general means and the global frameworks needed to study local or contextual phenomena.
Once local universes or structures for a language have been introduced, as they have with Tarski, the category of all structures for a given formal language is a natural place to consider.

This ties in with a general philosophy, emphasized by Grothendieck in his "Eléments de Géométrie Algébrique" ([EGA]), that one should try to develop all concepts of algebraic geometry in a relative context. Instead of always working over a fixed base field, and considering properties of one variety at a time, one should consider a morphism of schemes $f : X \to S$, and study properties of the morphism. It then becomes important to study the behavior of properties of f under base extension, and in particular, to relate properties of f to properties of the fibres of f.
(R. Hartshorne, Algebraic Geometry, p. 89.)

Even though I will not argue for this, it is fair to say that category theory has been instrumental in developping concepts "in a relative context" (e.g., descent theory).

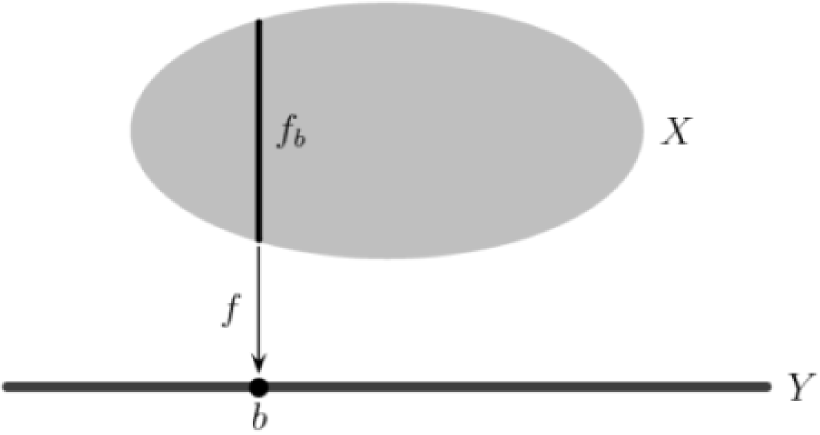
In what follows, I will refer to the category of all structures for a given first-order language, or to the category of all models of a given first-order theory.

On the basis of that category, I shall explore how fibered categories can be introduced:

- to develop Tarski's interpretation of variables ”in a relative context”
- to cast a new light on basic notions of model theory
- to put forward a categorical notion of contextuality within logic.

Fibrations and indexed categories

Fibration = topological and categorical generalization of a surjective map.



Given a category B, an *indexed category* over B is a pseudofunctor $F : B^o \to Cat$. So it is a correspondence:

- sending each object b of B to a category $F(b)$, called the "fiber" above b.
- sending each arrow $u : b \to b'$ in B to a functor $F(u) : F(b') \to F(b)$, also written u^*.

(As a pseudofunctor, F also involves natural isomorphisms $\mathrm{id} \Rightarrow \mathrm{id}^*$ and $u^* v^* \Rightarrow (vu)^*$ subjected to certain coherence conditions.)

To simplify things, an indexed category amounts to a family of categories that is organized in a functorial way.

Given an indexed category F,

- B is the "base category"
- $E := \coprod_{b \in \mathrm{Ob}\ B} F(b)$ is the "total space"
- the induced functor $\pi_F : E \to B$ that sends any object in $F(b)$ to b, any object in $F(b')$ to b', and so on, is a *fibration*.

Conversely, any fibration $\pi : E \to B$ gives rise to the indexed category $F_\pi : b \mapsto E_b := \pi^{-1}(b)$.

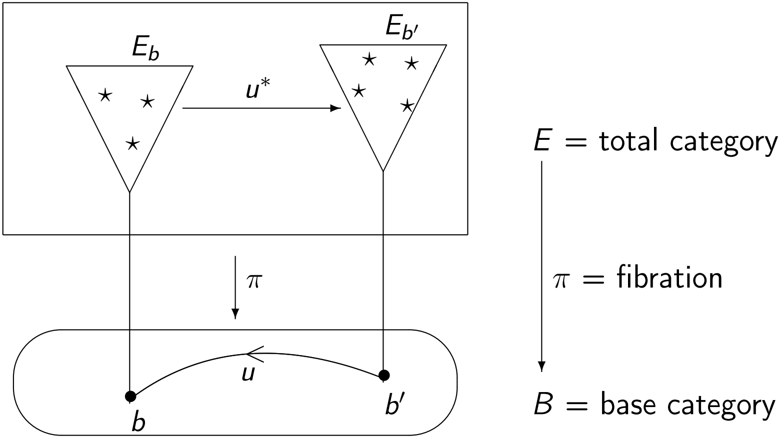
The collection of all objects x of E lying above b (i.e., $\pi(x) = b$) and the collection of all "vertical" arrows f in E above b (i.e., $\pi(f) = 1_b$) make up a category E_b, the *fiber* above b.

A fibration amounts to an indexed category, but free of the choice of any particular indexation.

Summary:

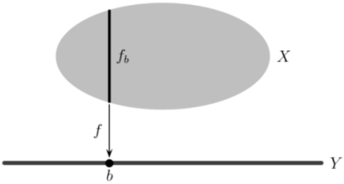
- $F : B^o \to Cat \quad \rightsquigarrow \quad \pi_F : \coprod_{b \text{ in } B} F(b) \to B$
- $\pi : E \to B \quad \rightsquigarrow \quad F_\pi = (\pi^{-1}(b))_{b \text{ in } B}$

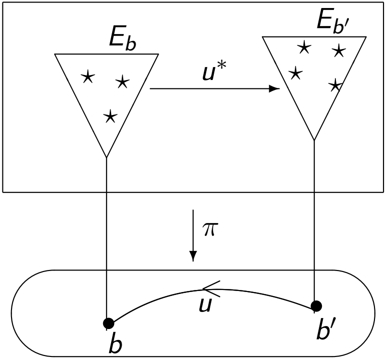
I will henceworth introduce indexed categories but present them as fibrations.



A fibration in general

Surjection vs Fibration





E = total category

π = fibration

B = base category

As a functor, an indexed category F sends any arrow in the base to a functor $F(u) = u^*$, called a "reindexing" functor between the corresponding fibers.

So the existence of a fibration requires then some *systematic connection* between the relations between any two points in the base (arrows u), and the relations between the corresponding fibers in the total space (functors u^*).

Owing to that systematic correspondence, one can get a sense here that the base category of a fibration works as a *control space*. This is how fibrations generalize surjections.

The base category can be endowed with any kind of structure and that the connections between the fibers must "lift" that structure and mirror it back to the total space.

To sum up, a fibration is a way to enrich a surjection with some structure: structure in the basis and structure both inside each fiber and between the fibers.

Now, my claim is:

- that the concept of fibration can be applied to Tarski's semantics;
- that the framework of fibrations is the natural framework to enrich Tarski's semantical scheme for variables with additional structure.

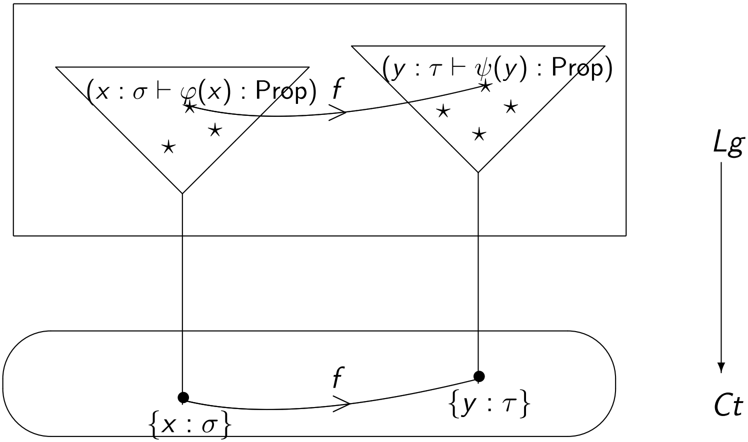
Tarskian semantics and fibered categories

The use of fibrations in logic is not unprecedented:

- Syntactic fibrations for type-theoretic systems
- Lawvere's fibration for quantification within a domain.

In the case of a syntactic fibration, the objects of the base category are logical contexts (declarations of variables) and the fiber above each context consists in all the logical judgments (sequents) that can be expressed in that context.

A logical calculus then is nothing else but a set of connections between judgments lying above (possibly) different contexts, i.e. a set of connections between fibers.



Conditions on f:

- $x : \sigma \vdash f(x) : \tau$
- $x : \sigma \parallel \varphi(x) \vdash \psi[f(x)/y]$

Syntactic fibration

In a syntactic fibration, structure comes from the logical rules. The admissible moves along the base category and, accordingly, along the total space, are constrained.
But this is syntactic stuff, and has nothing to do with Tarski's semantics.

Lawvere's fibration has to do with Tarskian semantics. But it is implicitly confined to a given domain.

My goal: describing Tarski's logical semantics as a fibration whose base is made up by *all* the interpretations of a given first-order language.

The typical clause for the interpretation of quantification in Tarski's semantics is, for any first-order structure M and any assignment σ of values to the variables of the language:

$M \vDash \forall x \phi(x)[\sigma]$ iff, *for any assignment* σ' that differ with σ at most at x, $M \vDash \phi(x)[\sigma']$.

Variables of the language are replaced with variables of assignments.

Tarski's fibration

Base category $\mathcal{S}$:

- objects are all structures for a fixed first-order language L;
- there is an arrow $\bar{f} : N \to M$ in $\mathcal{S}$ each time there exists an L-homomorphism $f : M \to N$.

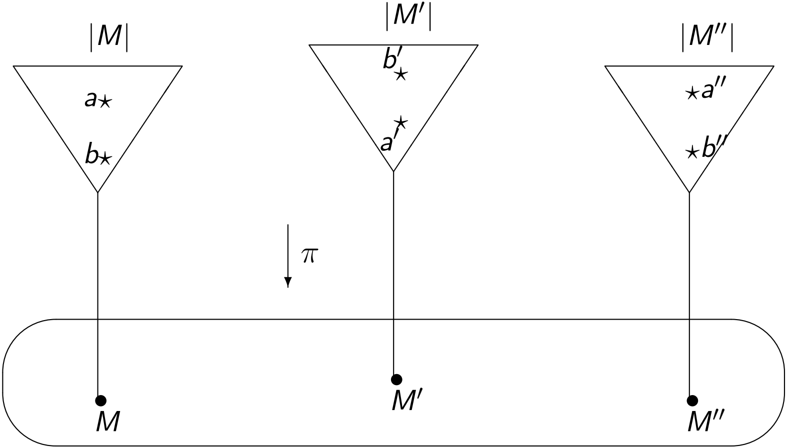
Pseudo-functor F:

- $F(M) = |M|^{\mathsf{Var}}$ (set of all assignations in M)
- $F(\bar{f}) = \bar{f}^* : \sigma \in |M|^{\mathsf{Var}} \mapsto f \circ \sigma \in |N|^{\mathsf{Var}}$.

This defines an indexed category $F : \mathcal{S}^o \to Cat$.

Hence a fibration $\pi : \mathcal{A} \to \mathcal{S}$, where $\mathcal{A}$ is the union of all $|M|^{\mathsf{Var}}$, $M \in S$.

π = **Tarski's fibration**.



Tarski's **fibration** π

Generalized assignments

In Tarskian semantics, given a *fixed* interpretation structure $M \in S$, each assignment gives a specific value in M to *each* variable symbol of L.

Let's turn things around: let's consider a fixed variable symbol 'x' and consider the value that it gets in *each* possible structure for the language.
An "x-generalized assignment" is nothing else but a choice function $\alpha \in \prod_{M \in S} |M|$.

A *generalized assignment* does the same thing, but for all variables. It is a map that assigns to each structure $M \in S$ and each variable symbol $v \in \mathrm{Var}$ a given element of the domain of M.

A generalized assignment is nothing else but a **section** of Tarski's fibration.
Section of a fibration = successive choice of a specific object in *each* fiber.

A distinction has to be made between Tarski's *fibration* and Tarski's *semantics*.

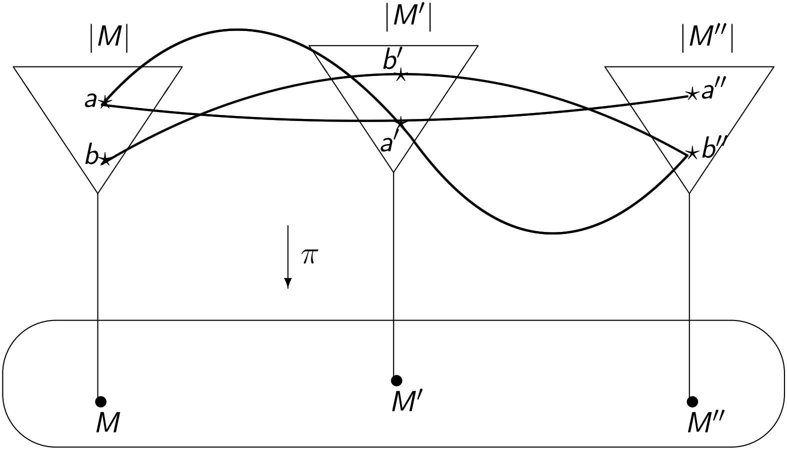
According to Tarski's fibration, only certain generalized assignments are admissible.
According to Tarski's semantics, all generalized assignments are admissible.

All the different values that an x-generalized assignment α gives to 'x' can be conceived of as a sequence of *counterparts* of an original value.

For instance, if $\alpha(M) = a$, $\alpha(M') = a'$ and $\alpha(M'') = a''$, the sequence a', a'', $\ldots$ can be viewed as what a becomes when one shifts from context $|M|$ to context $|M'|$, then to context $|M''|$, and so on.

According to Tarski's *fibration*, such a sequence is allowed on the condition that $a' = f(a)$ for some L-homomorphism $f : M \to M'$ and that $a'' = g(a')$ for some L-homomorphism $g : M' \to M''$. Stronger conditions could be set if L-homomorphisms were replaced, for instance, with elementary embeddings.

On the contrary, according to Tarski's *semantics*, there need not be any connection between a, a' and a'', because the choice carried out by α is entirely free.
Tarski semantics corresponds to the limit case where *any map* $f : M \to N$ whatsoever is taken to induce an arrow $\overline{f} : N \to M$ in $\mathcal{S}$.
The base category $\mathcal{S}$ does not work as a real control space any more.



Tarski's **semantics**: all sections are allowed.

Structure comes from the way in which a total range of values (a "total space") is organized by prescribed sections.

The smaller the leeway in the choice of single values to build a section, the more constrained are the generalized values, and the more *structured* is the fibration.

Tarski's semantics deprives Tarski's fibration of any structure: no constraint is placed on sections of Tarski's fibration.

Dynamic logic

Dynamic logic conceives of assignments as being possible worlds. Existential quantification over x becomes a possibility operator:

$M, \sigma \vDash \Diamond_x \varphi$ iff there exists θ such that $\sigma R_x \theta$ and $M, \theta \vDash \varphi$.

Then it is possible to consider that the assignments which are genuinely accessible from a given assignment make up a proper subset of $|M|^{\mathsf{Var}}$ only.

J. van Benthem, *Exploring Logical Dynamics* (1996):
A *generalized assignment model* is a couple $\langle M, \mathbb{V} \rangle$, where M is a regular L-structure and $\mathbb{V}$ is a selection of "available assignments" in M.
First-order evaluation then goes:

$\langle M, \mathbb{V} \rangle, \alpha \vDash \exists x \phi$ iff, for some $d \in |M|$, $\alpha^x_d \in \mathbb{V}$ s.t. $\langle M, \mathbb{V} \rangle, \alpha^x_d \vDash \phi$.

The selection $\mathbb{V}$ can be used to reflect *dependencies* between variables. For example, one can force 'x' and 'y' to have always different values.

J. van Benthem, *Exploring Logical Dynamics*, p. 177:

> *Standard models are then 'degenerate cases' where all dependencies between variables have been suppressed.*

J. van Benthem & N. Alechina, "Modal quantification over structured domain" (1997):

> *The Tarskian truth condition for the existential quantifier reads as follows:*
>
> $$M, [\overline{e}/\overline{y}] \vDash \exists x \varphi(x, \overline{y}) \quad \Leftrightarrow \quad \exists d \in D : M, [d/x, \overline{e}/\overline{y}] \vDash \varphi(x, \overline{y})$$
>
> *This may be viewed as a special case of a more general schema, when the element d is required in addition to stand in some relation R to $\overline{e}$ – where R is a finitary relation* **structuring** *the individual domain D:*
>
> $$M, [\overline{e}/\overline{y}] \vDash \Diamond_x \varphi(x, \overline{y}) \textit{ iff } \exists d \in D : R(d, \overline{e}) \& M, [d/x, \overline{e}/\overline{y}] \vDash \varphi(x, \overline{y})$$
>
> *[...] One might read $R(d, \overline{e})$ as*
>
> - *d can be constructed using $\overline{e}$,*
> - *d is not "too far" from the e's,*
> - *after you have picked up e's from the domain without replacing them, d is still available,*
>
> *et cetera.*

van Benthem & Alechina, continued:

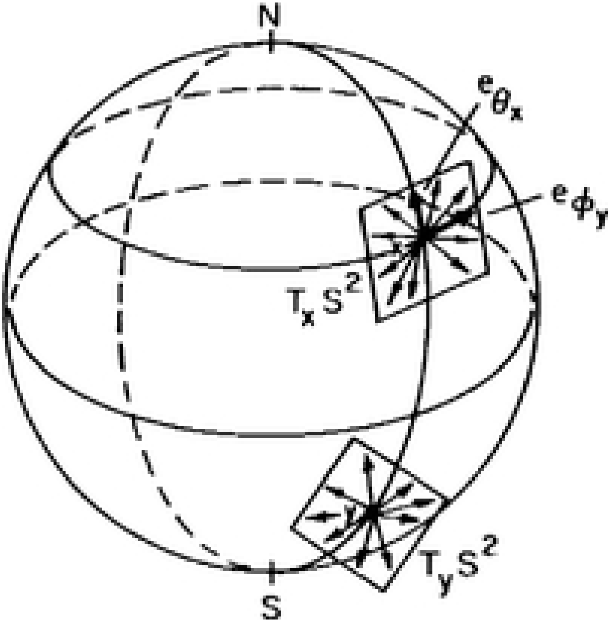
> *Ordinary predicate logic then becomes the special case of* **flat** *individual domains admitting of "random access", whose R is the universal relation.*

Significantly, van Benthem and Alechina spontaneously use geometric metaphors ("structuring", "flat"). But they consider only one structure M at a time.

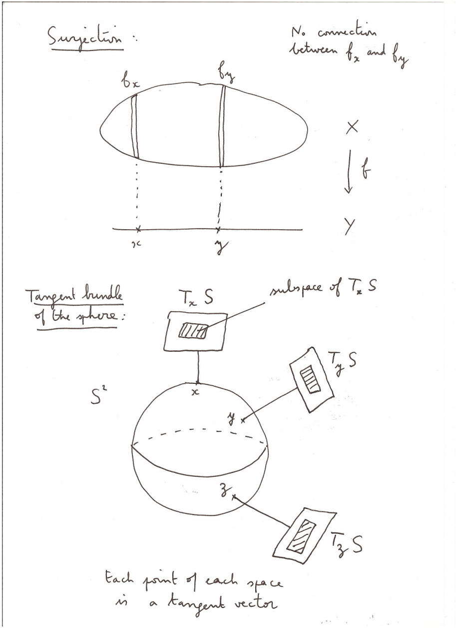
It makes more (philosophical and mathematical) sense to consider the whole base category $\mathcal{S}$.

In fact, a selection of "available assignments" for *each* object in $\mathcal{S}$ coincides with a kind of *distribution* on $\mathcal{S}$.

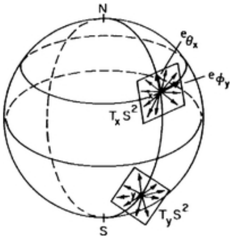
Tangent planes to the sphere:



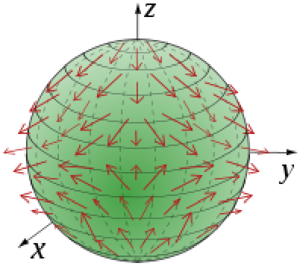
Tangent bundle and Distribution:



Tangent planes to the sphere:



Vector field on the sphere:



If one chooses one vector in each tangent space to the sphere, one gets a *distribution* which is called a *vector field*.

Fixing a vector field puts a maximal constraint upon all the admissible sections.

It also puts a constraint upon all the admissible curves on the base category (integral lines of the vector field).

Hence a distribution is a way to build structural constraints into the movements on the manifold at stake. It is the convenient way of stating a law in modern mechanics.

Dynamic logic can be looked at as a way of structuring Tarski's fibration in the way in which a distribution adds structure to a manifold.
In comparison, Tarski's variables appear as "flat" variables.

Non-logical constants as indexicals within a space of structures

W. Hodges, "Truth in a Structure" (1986), p. 148-150: non-logical constants can be given a meaning by being applied to a particular structure, exactly as the word 'yesterday' can be given a meaning by being used on a particular day.

> *On the present view, non-logical constants thus behave rather like the indexicals of a natural language, with the space of structures for the language of the theory playing a role analogous to the role played by space-time in fixing the reference of indexical expressions like 'yesterday' and 'today'. Just as the reference of such an indexical is determined relative to a spatio-temporal context, the reference of a non-logical constant is determined once we are given a structure for the language.*
> *(W. Demopoulos, "Frege, Hilbert, and the Conceptual Structure of Model Theory", p. 215)*

Sharpening Tarski's fibration

Structure can also be given to the base category $\mathcal{S}$.
Let $\mathcal{S}_T$ be the category:

- whose objects are of the models of some first-order theory T,
- whose arrows are all *elementary extensions* reversed:
  $\bar{f} : M_1 \to M$ iff $M \overset{\prec}{\underset{f}{\rightarrowtail}} M_1$.

Besides, to simplify matters, let's say that any formula to be considered has less than n_0 free variables (with n_0 big enough). So each fiber $|M|^{\mathrm{Var}}$ now becomes identical with $|M|^{n_0}$.

Resulting fibration: π_T (distinct from Tarski's fibration π).

Then, any formula $\varphi(x_1, \ldots x_{n_0})$ of L induces a distribution: for each M, the subset of $|M|^{n_0}$ made up of all the tuples that satisfy φ in M.

For any (partial) section α of π_T, the *generalized satisfaction* of a formula φ by α is readily definable:

$\alpha \vDash_g \varphi(\vec{x})$ iff, for any $M \in \mathrm{dom}(\alpha)$, $M \vDash \varphi(\vec{x})$ $[\alpha(M)]$.

Sections of π_T are restricted to partial sections along elementary chains $M_0 \prec M_1 \prec \ldots M_k \prec \ldots$

Moreover, a section α of π_T along an elementary chain $M_0 \xrightarrow[f_0]{\prec} M_1 \xrightarrow[f_1]{\prec} \ldots \xrightarrow[f_{k-1}]{\prec} M_k$ has to verify:

$\alpha(M_{i+1}) = \bar{f}_i^{\,*}(\alpha(M_i))$ for each $i \geq 0$.

Thus, for any section α of π_T, one has, for any formula φ:

$$\text{either } \alpha \vDash_g \varphi \text{ or } \alpha \vDash_g \neg\varphi$$

Let $\sim$ be the "indiscernibility relation" between n_0-tuples:

$\vec{a} \sim \vec{a'}$ iff $\vec{a}$ and $\vec{a'}$ belong to the same model M and satisfy exactly the same formulas in M.

Then, let's define:

- for any model M of T, let $G(M)$ be $\tilde{M} := |M|^{n_0}/\sim$
- for any $\bar{f}$ in $\mathcal{S}_T$ (corresponding to some elementary embedding $f : M \to M_1$) let $G(f) = \bar{f}^*$ be the functor that sends any indiscernibility class $[\vec{a}]$ in $\tilde{M}$ to $[f(\vec{a})]$ in $\tilde{M}_1$.

The functor G is an indexed category.

So, writing $\tilde{A}$ for the disjoint union of all $\tilde{M}$'s, G corresponds to a **new fibration** $\tilde{\pi_T} : \tilde{A} \to \mathcal{S}_T$.
In fact, $\tilde{\pi_T}$ is nothing but the fibration that one gets from π_T by "passing to the quotient":

$$\pi_T = \tilde{\pi_T} \circ \sigma_T$$

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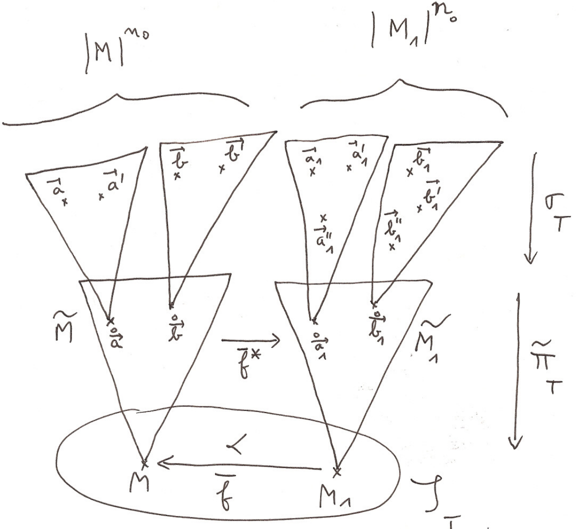

Types

Given a language L and an L-structure M, any maximally consistent set of formulas in L(M) with exactly one free variable x is called a *type*.
A type is *isolated* iff it consists in all the formulas in x that can be derived in T from a single formula $\varphi(x)$.

A type over M can be thought of as the description of an *ideal element* relatively to M.

In some cases, that ideal element is already realized in M, in some other cases it is not. If a model realizing all its ideal elements is said to be *saturated*.

The definition above carries over to the case of formulas with n_0 variables. So one gets n_0-types. The set of all n_0-types over M is written $S_{n_0}(M)$.

Remark

A pure n_0-type in $S_{n_0}(T)$ is nothing but a partial section of $\tilde{\pi}_T$.

Given p in $S_{n_0}(M)$, a chain c in $\mathcal{S}_T$ starting from M and a tuple $\vec{a}$ in $|M|^{n_0}$ realizing p in M, there is a unique section $\tilde{c}$ of π_T lifting c.

The section $\tilde{c}$ can be referred to as the "p-lift of c passing through $\vec{a}$."

Proposition

A given model M is saturated iff, for any type p, any elementary chain starting from M in $\mathcal{S}_T$ can be p-lifted.

The fibred space induced by a single model M

Let M be a fixed saturated model. For any (complete) n_0-type p, $p^M = \{\vec{a} \in |M|^{n_0} : M \vDash p[\vec{a}]\}$.

The disjoint union $\coprod_{p \in S_{n_0}(M)} p^M = |M|^{n_0}$ can be viewed as lying above $S_{n_0}(M)$.
Indeed, one has **a map** $\tau_M : |M|^{n_0} \to S_{n_0}(M)$, that assigns each $\vec{a} \in p^M$ to p.

Besides, $S_{n_0}(M)$ constitutes a topological space, with $U_\varphi := \{p : \varphi \in p\}$ as basic opens (for all formulas φ with exactly $x_1, \ldots x_{n_0}$ as free variables).

Moreover, $|M|^{n_0}$ can be endowed with the topology generated by all the definable subsets $\varphi^M(x_1, \ldots, x_{n_0})$ of $|M|^{n_0}$.

A *fiber bundle* is a continuous surjective map $f : E \to B$ between topological spaces such that, for any $x \in B$, the bundle of all the corresponding fibers $f^{-1}(y)$ in a neighborhood U of x make up a kind of smooth cylinder above x:

$$\begin{array}{ccc} f^{-1}(U) & \xrightarrow{\simeq} & U \times F \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle \mathrm{pr}_1} & \\ U & & \end{array}$$

$F =$ pattern common to all the fibers.

If the same group G acts upon each fiber, one gets a "G-principal bundle".

Theorem

Let M be countable and saturated. Then, for any two tuples $\vec{a}$ and $\vec{b}$ in M that have the same type over M, there is an automorphism f of M such that $f(\vec{a}) = \vec{b}$.

Corollary

The group $Aut(M)$ of all automorphisms of M acts freely and transitively on each fiber p^M.

Proposition

A countable model M is saturated iff τ_M is a G-principal bundle with $G = Aut(M)$.

Stable theories

For $M \prec M_1$, a type $p_1 \in S_{n_0}(M_1)$ is an *heir* of another type $p \in S_{n_0}(M)$ over M_1 iff

(i) the restriction of p_1 to $\mathrm{L}(M)$ is p and
(ii) for each formula $\varphi(\vec{x}, \vec{a}, \vec{b}) \in p_1$, with $\vec{a} \in M$ and $\vec{b} \in M_1$, $\varphi(\vec{x}, \vec{a}, \vec{a'}) \in p$ for some $\vec{a'} \in M$.

Any situation, with parameters in M, exhibited by p_1 over M_1 already has an instance proffered by p over M.

A theory T is *stable* iff any type in any model M of T has exactly one heir over any elementary extension of M.

Let T be a stable theory and let $\mathcal{S}_T$ be as above.

Then, let's define:

- $H(M) = \tilde{M}$
- for each $\bar{f} : M_1 \to M$ in $\mathcal{S}_T$, $H(\bar{f}) = \bar{f}^* : \tilde{M} \to \tilde{M_1}$ sends each realized type p in $S_{n_0}(M)$ to its unique heir along f in $S_{n_0}(M_1)$.

The functor H is an indexed category, whose base category is $\mathcal{S}_T$.

Hence a **corresponding fibration** $\lambda_T : \coprod_{M \in \text{Ob } \mathcal{S}_T} S_{n_0}(M) \to \mathcal{S}_T$.

Proposition

T is stable iff λ_T is well defined.

LOGIC	GEOMETRY
Mod T with homomorphisms as arrows	Tarski's fibration π
generalized assignments (Benthem & al.)	"distribution" over π
Mod T with elementary embeddings as arrows	π_T and quotient fibration $\tilde{\pi}_T$
elementary chains	privileged paths in Mod T
types in $S_{n_0}(M)$	partial sections of $\tilde{\pi}_T$
M saturated	map τ_M
M countable and saturated	τ_M is a fiber bundle
T stable	fibration λ_T

Summary of the correspondences