Models As Universes

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The problem addressed in this talk:

Is any logical consequence of ZFC ensured to be true?

This is the problem of the veridicity of ZFC, hereafter **the Veridicity Problem**.

Boolos-Kreisel problem: Kreisel ("Informal rigour"), Boolos ("Nominalist Platonism")

Given the language *L* of ZFC, how can we be sure that any logically valid *L*-sentence is true?

By itself, first-order logic does not seem to guarantee it at all.

Kreisel's answer is positive and appeals to the completeness theorem for first-order logic.

Boolos provides two answers, which resort to the reflection principle and to the completeness theorem, respectively.

The Boolos-Kreisel problem has been set up by Kreisel, and by Boolos as well, at the level of the background set-theoretic universe, namely:

Is any *L*-sentence that is logically valid (i.e. true in any structure contained in the universe), true in the universe?

This option lays itself open to the following attack: Logical validity w.r.t. the universe makes perfect sense, but truth in the universe cannot be defined explicity.

"Model-scaled view" = semantical view that considers only *L*-structures or models (as opposed to the background universe).

In this view, it makes sense to say that an *L*-sentence is true in some *L*-structure, but it seems to make no sense at all to say that that sentence is logically valid w.r.t. some *L*-structure.

Predicament:

	Kreisel-Boolos View	Model-Scaled View
Logical Validity	OK	?
Truth	?	OK

Ways of framing the Boolos-Kreisel problem

Kreisel's and Boolos' respective answers to the Boolos-Kreisel problem can be transposed and completed so as to provide answers to the Veridicity Problem.

Summary of the talk

- Kreisel's and Boolos' solutions adapted to the Veridicity Problem.
- Any model of set theory can be compared to a background universe and shown to contain "internal models." Implications for the Veridicity Problem.
- Further results about internal models.
- "Internal modal logic:" Internal models as accessible worlds.

Kreisel's answer adapted to the Veridicity Problem

For any sentence ϕ of *L*,

ZFC $\models^{+} \phi$:= "ϕ is true in any set or class structure that satisfies ZFC"
(this is the analog of Kreisel's Val ϕ).

$\alpha_{\in}$:= "α is true when the quantifiers in α range over all sets and $\in$ is taken to be the real membership relation"

Kreisel-style answer

ZFC $\models \phi$ entails ZFC $\vdash \phi$, which entails ZFC $\models^{+} \phi$, which entails in turn ϕ_{ϵ}: problem solved.

Main difficulty:
One is led to consider that a sentence is true in this a *little bit special* model that is the universe of all sets.

But the status of truth in the universe is problematic.

Iterative conception: The universe can be regarded only as a potential totality, and, as a consequence, truth in the universe should not be regarded as determined for every sentence.

Even though one is not willing to endorse the iterative conception, truth in the universe cannot be handled exactly in the same way as truth in a given model, since, as a matter of principle, no formal semantics can underpin both kinds of truth/

(Unless the universe is taken to be an actual model and plunged with all other models into some further background universe — but then, precisely, it would cease to be *the* universe.)

Kreisel's original answer can be sharpened:

PA $\vdash \ulcorner \vdash \phi \urcorner \to \phi$ (Mostowski theorem)
so ZFC $\vdash \ulcorner \vdash \phi \urcorner \to \phi$
But ZFC $\vdash \ulcorner \models \phi \urcorner \to \ulcorner \vdash \phi \urcorner$ (the completeness theorem is a theorem of ZFC)
so ZFC $\vdash \ulcorner \models \phi \urcorner \to \phi$.
Hence: "If ϕ is true in every model, then ϕ."
"Every sentence true in every model is true."

BUT of course truth in every model of ZFC does not amount to truth in every model whatsoever.

Ensuing from Löb's theorem:
$\texttt{ZFC} \vdash \ulcorner \models_{\texttt{ZFC}} \phi \urcorner \rightarrow \phi$ only if $\texttt{ZFC} \vdash \phi$.

"If ϕ is true in every model of ZFC, then ϕ"
can be derived only if ϕ is already a theorem of ZFC.

Another solution has to be found.

As opposed to the difficulties that affect the Kreiselian option, there is in fact a structure in which all the sentences of the language of ZFC are ensured to have formalized truth conditions and in which all the sentences derivable in ZFC are ensured to be true: namely, a model of ZFC!

That will be the starting point of the solution proposed in this talk, the main problem of which being to justify to see such a model as a genuine universe.

Boolos' answer adapted

Boolos as well remarks that, oddly enough, logical validity does not guarantee truth. He suggests two ways out:

1. the definition of "supervalidity" (as expressed by a monadic second-order sentence whose quantifiers are to be interpreted plurally) and the use of the completeness theorem.
   BUT there is no clear way of defining the notion of being a "superconsequence of ZFC."
2. the reflection principle — **BUT** about finite conjunctions only.

Solution: Extension of the reflection principle through the addition of a satisfaction predicate $\mathrm{Sat}(u, v)$ and a truth predicate $\mathrm{Tr}(u)$ to the language L of ZFC.

Assume a usual set-theoretic coding of syntax.
For any formula of L, let $\ulcorner\phi\urcorner$ the set that codes ϕ.

Form(x):= "x is (the code of) a formula"
Sent(x):= "x is a sentence"
Ax(x):= "x is an axiom of ZFC"
Assign(y):= "y is a map with domain the set of (all codes for) the variable symbols"

Sat($\ulcorner\phi\urcorner, s$) is intended to mean "$s$ is an assignment for the variables of L which satisfies ϕ in V"

The formulas ϕ for which one has Sat($\ulcorner\phi\urcorner, s$) should be the original formulas of L not containing 'Sat', so that no paradox arises.

Axioms for Sat:

1. $\forall x \forall y(\mathrm{Sat}(x, y) \rightarrow \mathrm{Form}(x) \wedge \mathrm{Assign}(y))$;
2. the usual inductive clauses for satisfaction:
   - $\mathrm{Sat}(v_1 \ulcorner \in \urcorner v_2, s) \leftrightarrow s(v_1) \in s(v_2)$
   - $\mathrm{Sat}(v_1 \ulcorner = \urcorner v_2, s) \leftrightarrow s(v_1) = s(v_2)$
   - $\mathrm{Sat}(\ulcorner \neg \urcorner u, s) \leftrightarrow \neg \mathrm{Sat}(u, s)$,
     $\mathrm{Sat}(u \ulcorner \vee \urcorner u', s) \leftrightarrow (\mathrm{Sat}(u, s) \vee \mathrm{Sat}(u', s))$
   - $\mathrm{Sat}((\ulcorner \exists \urcorner v)u, s) \leftrightarrow \exists x \mathrm{Sat}(u, s[x/s(v)])$
3. $\mathrm{Tr}(u) \leftrightarrow (\mathrm{Sent}(u) \wedge \forall y(\mathrm{Assign}(y) \rightarrow \mathrm{Sat}(u, y)))$.

Let ZFCS be the resulting system in $L^{+} = L \cup \{\mathrm{Sat}, \mathrm{Tr}\}$, where the replacement axiom and the separation axiom are extended to include formulas in which 'Sat' or 'Tr' occurs.

It is well-known that semantic notions about L can be formalized within L. This formalization readily extends to L^+.

In particular, there is a formula $\Sigma(A, u, s)$ in L^+ to the effect that A is a set structure for L^+, u is $\ulcorner \phi \urcorner$ for some formula ϕ of L^+ and ϕ holds in A under the assignment s.

Accordingly, there is a formula $\Theta(A, u) = \ulcorner A \vDash \sigma \urcorner$ in L^+ to the effect that A is a structure for L^+, u is $\ulcorner \sigma \urcorner$ for some sentence σ of L^+, and $A \vDash \sigma$.

One thus deals with two truth predicates, Tr and Θ.

The proof of the reflection principle for ZFC extends readily to ZFCS.

Now, suppose that ϕ is not true. Then ZFCS proves that $\neg\phi$ is true and thus (by reflection) that $V_\beta \models \text{ZFC} + \neg\phi$ for some β, and so ϕ is not a logical consequence of ZFC.

So ZFCS proves any logical consequence of ZFC to be true (in the sense of 'Tr', which has been defined in L^+ but is not definable in L, owing to Tarski's theorem on the undefinability of truth).

BUT ZFCS is significantly stronger than ZFC, since, as just shown, it proves Con(ZFC). One should argue just from within ZFC.

Summary:

	Kreisel's Modified View	Boolos' Modified View
Truth in the universe	Informal	Formalized through a satisfaction predicate added to the language of ZFC
Answer to the KST problem	Trivialized by Löb's theorem	Requires to shift to ZFCS, a proper extension of ZFC

The Veridicity Problem: Is any logical consequence of ZFC true?

$\mathbf{ZFCS} \vDash \ulcorner \mathbf{ZFC} \vDash \phi \urcorner \rightarrow \mathbf{Tr}(\ulcorner \phi \urcorner)$

The question naturally arises as to whether such a logical consequence of ZFCS is true itself. The answer to the Veridicity Problem has just been pushed back up a level.

The natural way to go is to frame the Veridicity Problem at the level of models of ZFC, so that any definition of truth in the universe becomes unnecessary.

Obviously, the counterpart of that option is the need to define what it means for a sentence of L to be, relatively to some model of ZFC, a logical consequence of ZFC.

Boolos and Kreisel considered two kinds of truth (truth in a set structure and truth in the background universe).

It is clearer to deal with only one kind of truth. The notion of truth that occurs in the definition of being a logical consequence of ZFC, as truth in any structure for the language, should be the same as that about which it is asked whether or not it is ensured by being a logical consequence of ZFC.

Confining oneself to the model theory of models of ZFC avoids any recourse to a more robust background theory.

Both Kreisel and Boolos tended to consider the background set-theoretic universe as a kind of monster model (the intended model of the set metatheory).

Let's turn things around, by turning *each* model of ZFC into a universe in its own right.

Such a model-scaled construal of the Veridicity Problem is actually compatible with the "Multiverse View," yet does not force its endorsement.

The Multiverse View holds that there are a multitude of set-theoretic universes, each of which embodies a concept of set and a set-theoretic truth of its own.
The Multiverse View amounts to suggesting that there are as many universes as there are models of ZFC.

The Model Scaled View which I advocate consists in identifying all models of ZFC with as many universes.

Standard coding. The code of any formula ϕ of L consists in a sequence $\ulcorner\phi\urcorner$ of numerals, and gives rise in any model M of ZFC to an interpretation $\ulcorner\phi\urcorner^M$, where each numeral of the sequence is interpreted by the corresponding integer of M.

The main notions in the metatheory of ZFC ("Being a formula," "Being a proof in ZFC," "Being a model of a sentence") can be formalized within the first-order language L of ZFC.
For instance, it is possible to define in L the predicate 'For(x)' to the effect that x encodes the construction of a formula of L.

An M-formula is an object a in $|M|$ such that $M \vDash$ For$(x)[a]$.
Any $\ulcorner\phi\urcorner^M$ is an M-formula, but the converse is not true.

A *model* $M = \langle M, \in_M \rangle$ of ZFC is ω***-standard*** if $\in_M$ is transitive and well-orders all the finite ordinals of M.
If M is ω-standard, the M-formulas (resp. the M-proofs) are in a 1-1 correspondence with the genuine formulas (resp. the proofs) of ZFC. If not, some M-formulas and M-proofs fail to correspond to any formula or proof of ZFC.

Let M be a model of ZFC, and N an element of $|M|$ such that $M \vDash \ulcorner$"N is an interpretation structure for L"$\urcorner$.

This implies that there exists $|N|, E^N \in |M|$ such that $M \vDash (N = \langle |N|, E^N \rangle \wedge E^N \subseteq |N| \times |N|)$.

One then defines:

$$|N_M| := \{x \in |M| : M \vDash x \in |N|\}$$

$$E^N_M := \{(x, y) \in |N_M| \times |N_M| : M \vDash (x, y) \in E^N\}$$

The structure $N_M := \langle |N_M|, E^N_M \rangle$ is called the *replica of N in M*.

Lemma

For any sentence ϕ of L and any model M of $\texttt{ZFC}$*, one has:*
$M \vDash \ulcorner N \vDash \phi \urcorner$ *iff* $N_M \vDash \phi$.

Theorem (Suzuki-Wilmers (1971), Schlipf (1978))

Let M be a model of $\texttt{ZFC}$*. Then there exists* $N \in |M|$ *such that*
$N_M \vDash \texttt{ZFC}$ *(but not necessarily:* $M \vDash \ulcorner N \vDash \texttt{ZFC} \urcorner$*).*

Proof. Two cases:

- M is ω-standard.
  The very existence of M implies that ZFC is consistent (so, no proof of '$0 = 1$'). By hypothesis, there are no more proofs according to M than there are in reality, so $M \models \mathrm{Con}(\text{ZFC})$. And since the completeness theorem is true in M... (use the lemma).
- M is not ω-standard. The idea is to index all the axioms of ZFC by some nonstandard ordinal of M, so that the reflection principle can be applied to what M thinks to be a finite conjunction of axioms.
  Suppose $M \models \neg\exists\alpha \ulcorner V_\alpha \models \text{ZFC}\urcorner$. Given $(A_i)_{i\in\mathbb{N}}$ a recursive enumeration of the axioms of ZFC, one gets, by compactness: $M \not\models \forall n \exists\alpha \ulcorner V_\alpha \models A_0 \wedge A_1 \wedge \ldots \wedge A_n\urcorner$. Given the L-formula $\chi(\underline{n}) := \exists\alpha \ulcorner V_\alpha \models A_0 \wedge A_1 \wedge \ldots \wedge A_n\urcorner$, there exists $n_0 \in \omega^M$ such that $M \models \neg\chi(n_0) \wedge \forall n < n_0\ \chi(n)$,.
  Owing to the reflection principle, n_0 has to be a nonstandard integer of M. But $M \models \chi(n_0 - 1)$, and $n_0 - 1$ is also nonstandard.

An *internal model* of ZFC is any model of ZFC of the form N_M, for some model M of ZFC .

The previous result ensures that any model M of ZFC has internal models. Hence it becomes possible to define logical consequence from ZFC **w.r.t. some model** M.

Two special models:

- Shepherdson's minimal model M_0 of ZFC.
  All internal models of M_0 are nonstandard, and M_0 faithfully recognizes them to be so.
- Any model M^* of ZFC + $\neg$Con(ZFC).
  M^* does have internal models, but from the point of view of M^* they satisfy at most a finite number of the axioms of ZFC (only, this number is nonstandard).

More generally, one is justified in considering any model of set theory, not only as a **domain**, that is as a place of evaluation of formal sentences, but also as a ***point of view***, that is as a background universe on its own, which includes models and establishes a specific satisfaction relation between them and formulas.

This does not detract from the absolute point of view of the real universe, which is but the semantic counterpart of the fact that the analysis is kept within the limits of ZFC.

Viewing models as "points of view" only catches up with a well-established tradition dating back to Skolem's paradox. Any member a of a model M of ZFC gives rise to the set $a^* = \{x \in |M| : x \in_M a\}$. The set a^* (in V) *is* nothing but a as seen from the point of view of M, even though a^* does not necessarily belong to M. The relativity phenomenon in which Skolem's paradox is grounded is "the discrepancy between M's assessment of a and a's (or rather, a^*'s) true status." (I. Jané)

The notion of point of view itself corresponds to the set-theoretic operation $(M, N \in |M|) \mapsto N_M$.

To sum up. While Kreisel and Boolos referred to the universe as being a kind of model, any model of ZFC can be looked at as being a surrogate universe from the point of view of which other models of ZFC can be considered.

No violation of the axiom of foundation, because an internal model N_M does not necessarily coincide with the element N of $|M|$.
The progression is: starting with a model M_0, there is $M_1 \in |M|$ such that $M_1' := (M_1)_M \models \mathtt{ZFC}$.
Then there is $(M_2)_{M_1'}$ with M_2 belonging to M_1', but not necessarily to M_1 — so that any infinite descending $\in$-chain $\ldots |M_2| \in |M_1| \in |M|$ is avoided in the end.

Depth of logical consequence?

Definition

An L-sentence is a *2-logical consequence* of ZFC if it is true in any internal model of ZFC.

In fact, 2-logical consequences and logical consequences of ZFC turn out to collapse:

Proposition

Let ϕ be a sentence of L. Then ϕ is a 2-logical consequence of ZFC iff it is a logical consequence of ZFC.

Definition

Let ϕ be an L-sentence and M be a model of ZFC.
ϕ is an *M-logical consequence of* ZFC, written ZFC $\models_M \phi$, iff, for every $N \in |M|$, $N_M \models$ ZFC implies $N_M \models \phi$.

The intuitive meaning of ZFC $\models_M \phi$ is that ϕ would be a logical consequence of ZFC *were M the background universe*.

Definition

ϕ is called an *internal logical consequence of* ZFC, written ZFC $\models^{\mathrm{i}} \phi$, iff ZFC $\models_M \phi$ for any model M of ZFC.

The intuitive meaning of ZFC $\models^{\mathrm{i}} \phi$, then, is that ϕ is a logical consequence of ZFC from the points of view of all models.

Relationship between ZFC $\vDash_M \phi$ and $M \vDash \phi$

Let θ be the first strongly inaccessible ordinal. By a result of Montague and Vaught, there exists $\theta^* < \theta$ such that $\langle V_{\theta^*}, \in\rangle \equiv \langle V_\theta, \in\rangle$, and $(V_{\theta^*})_{V_\theta} = V_{\theta^*}$. Consequently, ZFC $\vDash_{V_\theta} \phi$ implies $V_\theta \vDash \phi$.

Let's call a cardinal γ a *universe cardinal* iff $V_\gamma \vDash$ ZFC, and let γ_0 be the least universe cardinal.
The *weak axiom of universes* is the sentence WAU of *L* saying that "there are unboundedly many universe cardinals." For κ inaccessible, $V_\kappa \vDash$ ZFC + WAU.
But, by minimality, $V_{\gamma_0} \nvDash$ WAU, and in fact $V_\kappa \vDash \ulcorner V_{\gamma_0} \nvDash \text{WAU} \urcorner$.

Consequently, $M \vDash \phi$ does not entail ZFC $\vDash_M \phi$.

The two conditions $\texttt{ZFC} \models_M \phi$ and $M \models \phi$ do not have the same meaning. Still, $\texttt{ZFC} \models_M \phi$ is true of each M iff $M \models \phi$ is.

Theorem

Let ϕ be an L-sentence. Then:

$$\texttt{ZFC} \models \phi \; \textit{ iff } \; \texttt{ZFC} \models^i \phi$$

Back to the Veridicity Problem

	Kreisel's Modified View	Boolos' Modified View	Model-Scaled View	Generalization to every M
ϕ is a logical consequence of ZFC	$\mathrm{ZFC} \vDash^{+} \phi$	$\mathrm{ZFC} \vDash \phi$	$\mathrm{ZFC} \vDash_{M} \phi$	$\mathrm{ZFC} \vDash^{\mathrm{i}} \phi$
ϕ is true	ϕ is informally true	$\ulcorner\phi\urcorner$ is in the extension of the truth predicate added to L	ϕ is true in M	ϕ is true in every M
Answer to the Veridicity Problem	Yes	Yes	No in general	Yes

The last column of the table above is but the generalization to every M of the model-scaled view relativized to some model M of ZFC (as expressed by the previous column).

Let's now focus on the consideration of the internal models of a single model M of $\textsc{zfc}$.

Remark

The class of all models internal to M is not definable over M.

(This is because n is a nonstandard integer of M if and only if whenever $M \models \ulcorner N \models$ the first n axioms of $\textsc{zfc}\urcorner$, N_M is a model of $\textsc{zfc}$. M cannot define its nonstandard integers.)

Proposition

Let M be a model of $\textsc{zfc}$. We define the *standard system of M* as being the set of the standard truncatures of all real numbers of M:

$\mathrm{St}(M) = \{\mathrm{st}(A) : A \in |M|, M \models A \subseteq \omega\}$, where

$\mathrm{st}(A) = \{n \in \mathbb{N} : M \models \overline{n} \in A\}$.

Then there is $N \in |M|$ such that $N_M \equiv M$ iff $\mathbf{Th}(M) \in \mathbf{St}(M)$.

In that case, being an M-logical consequence of $\textsc{zfc}$ ensures truth in M.

The criterion given by the proposition above really divides the spectrum of all models into two camps.

Indeed, any full standard model of second-order set theory contains every real, and hence in particular its own standard system.

On the other hand, the theory of any pointwise definable model M of ZFC cannot be in M's standard system.
(Otherwise, you can mimick the Liar paradox in M.)

Corollary

For any transitive $\in$-model $\langle M, \in \rangle$ of ZFC*, there is $x \in |M|$ such that $\langle x, \in \rangle \equiv \langle M, \in \rangle$ iff, for any M-definable subtheory S of $Th(M)$, $M \models \ulcorner$there is a standard model of $S\urcorner$.*

The natural step to take to strengthen the previous proposition is to require that the internal model is an *elementary* substructure of the original one.
Actually, the set of sentences true in (M, V_α^M) is too big to be a set in M. The best approximation of the existence of an internal elementary substructure is:

Proposition

Let M be a model of ZFC and α an ordinal of M.
Then there exists $N \in |M|$ such that

- $V_\alpha^M \subseteq |N|$ and
- $(N_M, V_\alpha^M) \equiv (M, V_\alpha^M)$

iff $\exists s : (V_\alpha^M)^{<\omega^M} \to \wp(\omega^M), s \in |M|$, such that $\forall \vec{a} \in V_\alpha^M \ \mathrm{st}(s(\vec{a})) = \mathrm{Th}(M, \vec{a})$.

Obviously, the minimal model M_0 does not have any internal model N_M as an elementary substructure.

On the contrary, any recursively saturated model of ZFC has one.
(Suppose that M is a recursively saturated model of ZFC.
Then consider the type composed of all formulas $\phi_n(x)$ = "any tuple of V_x satisfies any of the first n formulas of L in V_x exactly when it satisfies it in M".)

Stronger and weaker theories

The previous results can be extended to set theories stronger than ZFC, in particular to Morse-Kelley set theory (MK), which is a first-order two-sorted analog of second-order set theory.

Some results can also be found about set theories weaker than ZFC, in particular Kripke-Platek set theory with urelemente (KPU).

The fragment $\mathsf{L}_{\mathbb{A}}$ defined by an admissible set $\mathbb{A}$ is the set of all formulas φ of $\mathsf{L}_{\infty,\omega}$ whose codes belong to $\mathbb{A}$.
Classical result: If two structures $M, N \in |\mathbb{A}|$ are $\mathsf{L}_{\mathbb{A}}$-elementary equivalent, then they are potentially isomorphic (and thus isomorphic in case M and N are countable). Consequently, $\mathsf{L}_{\mathbb{A}}$ measures the variety of countable structures internal to $\mathbb{A}$, in the sense that if, $M, N \in |\mathbb{A}|$ are not isomorphic, then there exists a discriminating sentence $\phi \in \mathsf{L}_{\mathbb{A}}$ such that $M \models \phi$ and $N \models \neg\phi$.

Remark: Because admissible sets are supposed to be transitive $\in$-models of KPU, any admissible set $\mathbb{A} \in \mathbb{B}$ coincides with the corresponding internal model $\mathbb{A}_{\mathbb{B}}$.

Proposition

Let $\mathbb{A}$, $\mathbb{B}$ two admissible sets such that $\mathbb{B} \in |\mathbb{A}|$ and $\mathbb{A} \models \ulcorner\mathbb{B}$ is countable$\urcorner$. Then for any $\phi \in$ $L_{\mathbb{B}}$, one has that $\mathbb{A} \models \ulcorner\phi$ is valid$\urcorner$ iff $\mathbb{B} \models \exists P \ulcorner P$ is a proof of $\phi\urcorner$.

Corollary

Let $\mathbb{A}$ and $\mathbb{B}$ be admissible sets, with the same hypotheses as above. Then, for any sentence ϕ of $L_{\mathbb{B}}$, ZFC $\models_{\mathbb{A}} \phi$ iff ϕ is valid.

Idea: Going modal by thinking of any internal model N_M as being *a possible world accessible from* M.

Definition

M' is accessible from M iff M' is (isomorphic to) some model of ZFC internal to M.

The difference with Hamkins-Löwe's "modal logic of forcing" is that the accessibility relation works downward instead of going upward.

The spontaneous additional semantical clause is:

Definition

Let M be a model of ZFC and ϕ a sentence of the language L of ZFC.
$M \models \Diamond\phi$ iff there is $N \in |M|$ such that $N_M \models \text{ZFC} + \phi$.

But of course the operators cannot be added directly to the language L of ZFC.

L' = language of propositional modal logic.

An *interpretation* i of L' into L is a map that assigns to each propositional letter p an arbitrary sentence of L.

For any such interpretation i, any structure M for L and any modal formula A, '$M \models i(A)$' is defined inductively as follows:

- $M \models i(\neg A)$ iff $M \not\models i(A)$
- $M \models i((A \wedge B))$ iff $M \models i(A)$ and $M \models i(B)$
- $M \models i(\Box A)$ iff $\textsc{zfc} \models_M i(A)$.

Definition

Given a formula A of L' and a model M of ZFC,
A is *modal-internally valid in M* iff, for any interpretation i of L' into L, $M \models i(A)$.

Definition

A formula A of L' is a *valid principle of internal modal logic* if it is modal-internally valid in any model of ZFC.

The set of all valid principles of internal modal logic is denoted by IML.

Remark: There is no formula $P(x)$ of L such that, for any sentence ϕ of L and any model M of ZFC, $M \models \Diamond\phi$ iff $M \models P(\overline{n_\phi})$.

Internal modal logic also has an expressive power of its own in comparison to usual systems of modal logic. Indeed, propositional modal logic cannot easily make sense of the possibility operator being applied to a whole theory, instead of a single sentence.
($M \models \Diamond T$ iff there is $N \in |M|$ such that $N_M \models T$, for any theory T extending ZFC.)

It is straightforward to check that IML is a normal modal logic.

Besides, the axiom **T** is warranted by the same workings as the first theorem (with ZFC + ϕ replacing ZFC).
Thus, IML is a way of encoding the existence of internal models in the guise of a **T**-style axiom: it establishes a connection between set-theoretic reflection and modal reflexivity.

Lemma

$\neg GL \in$ IML.

Definition

A $\mathcal{K}$*-valid principle of internal modal logic*, for a given class $\mathcal{K}$ of models of ZFC, is a formula A of L' that is modal-internally valid in any member of $\mathcal{K}$.
This is written: $\mathcal{K} \models_{\text{IML}} A$.

Not every class of models can be considered. Indeed, it has to be stable under internal models.

Definition

So let's say a class $\mathcal{K}$ of models of ZFC is *weakly downward stable* if, for every $M \in \mathcal{K}$, there exists $N \in |M|$ such that $N_M \in \mathcal{K}$.

It is *strongly downward stable* if, for every $M \in \mathcal{K}$ and every $N \in |M|$, $N_M \models \text{ZFC}$ implies $N_M \in \mathcal{K}$.

Lemma

The class $\mathcal{S}$ of all standard models of ZFC *and the class $\mathcal{T}$ of all transitive models of* ZFC *are not weakly downward stable.*

Theorem

$\mathcal{S} \not\models_{\text{IML}} \mathbf{5}$.

Theorem

$\mathcal{T} \models_{\text{IML}} S4$.

Lemma

The class $\mathcal{R}$ of all countable recursively saturated models of `ZFC` *and the class $\mathcal{N}$ of all non-ω-standard models of* `ZFC` *are both strongly downward stable.*

Remark: The class $\mathcal{R}$ has been studied by Victoria Gitman and Joel Hamkins, and proved to be a model of “the multiverse axioms.”

Theorem

$\mathcal{R} \models_{\text{IML}}$ *S4.*

Remark: This result uses the equivalence between “countable recursively saturated” and “countable resplendent.”

Definition

A modal theory Λ is IML-*complete w.r.t. a class* $\mathcal{K}$ *of models of* ZFC if, for any formula A of L', $\Lambda \vdash A$ iff $M \vDash_{\text{IML}} A$ for every $M \in \mathcal{K}$.

Proposition

S4 is IML-complete w.r.t. $\mathcal{R}$.

Conclusion

1. Any model of set theory can be seen as a local universe, because it can be shown to embrace internal models. Not only truth in any given model of ZFC, but also logical consequence of ZFC w.r.t. any such model, make sense after all.
2. A model-scaled treatment of the Veridicity Problem has to be favored, because it does not resort to any informal notion of truth in the background universe, and does not exceed the limits of ZFC either.
3. Moreover, the study of internal models of ZFC, or "set-theoretic prospecting," provides fine-grained results —whether in modal terms or in purely set-theoretic ones—, and holds out hope of further results.

THANK YOU !