Homologic

Brice Halimi

June 5, 2010

Model theory is:

- universal algebra + logic (Chang and Keisler)
- the study of the construction and specification of structures within specified classes of structures (Hodges, *Model Theory*)
- algebraic geometry minus fields (Hodges, *A Shorter Model Theory*)
- the geography of tame mathematics (Hrushovski, quoted by van den Dries)
- geometrical categoricity theory (Zilber)

"There are, in fact, very few isomorphism theorems in real mathematics that depend only on cardinal invariants. A useful contribution of post-Morley model theory is to explain these extreme classifications in terms of a geometrical independence theory [...]" (Macintyre).

Formulas as *chains*:

$$\partial\varphi := \bigwedge_{i=0}^{n-1} \neg^i \forall x \varphi(v_0, \ldots, v_{i-1}, x, v_{i+1}, \ldots, v_{n-1})$$

For example:

- $\varphi(v_0, v_1, v_2)$ = starting point
- first, the conjunction of $\forall x\varphi(x, v_0, v_1)$, $\neg\forall x\varphi(v_0, x, v_1)$, and $\forall x\varphi(v_0, v_1, x)$;
- then, the conjunction of six formulas: $\forall y\forall x\varphi(x, y, v_0)$ and $\neg\forall y\forall x\varphi(x, v_0, y)$, $\forall y\neg\forall x\varphi(y, x, v_0)$ and $\neg\forall y\neg\forall x\varphi(v_0, x, y)$, $\forall y\forall x\varphi(y, v_0, x)$ and $\neg\forall y\forall x\varphi(v_0, y, x)$.

$$\partial \circ \partial \equiv \bot$$

Let L be a first order language containing at least a unary quantifier Q.
Let F_n be the set of formulas of L with exactly $v_0, \ldots, v_n$ as free variables. (F_{-1} may be defined as the set of all sentences of L.)
The two following applications $d_i : F_n \to F_{n-1}$ (for $n \geq 1$) and $s_i : F_n \to F_{n+1}$ can then be defined:

$$d_i(\phi(v_0, \ldots, v_n)) = Qx\ \phi(v_0, \ldots, v_{i-1}, x, v_i, \ldots, v_{n-1})$$

$$s_j(\phi(v_0, \ldots, v_n)) = ((v_j = v_{j+1}) \to \phi(v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_{n+1}))$$

As soon as, for every formula φ,

- $QyQx\varphi(y, x, \vec{u}) \equiv QyQx\varphi(x, y, \vec{u})$ (condition (a))
- $Qx((x = y) \rightarrow \varphi(y, \vec{u})) \equiv \varphi(x, \vec{u})$ (condition (b)),

the following *simplicial identities* are verified (up to logical equivalence):

- $d_i d_j = d_{j-1} d_i$ for $i < j$
- $s_i s_j = s_{j+1} s_i$ for $i < j$
- $d_i s_j = \begin{cases} s_{j-1} d_i & \text{for } i < j \\ \text{id} & \text{for } i = j \text{ or } i = j+1 \\ s_j d_{i-1} & \text{for } i > j+1 \end{cases}$

$F_*^Q = \langle F_n, (d_i^n)_{0 \leq i \leq n}, (s_j^n)_{0 \leq j \leq n} \rangle_{n \in \mathbb{N}}$ is a *simplicial set*.

"Formulas as chains":

- Any (generalized unary) quantifier Q satisfying conditions (a) and (b) as a "face operator"
- (s_j) as the corresponding sequence of "degeneracy operators"

What about connectives?

Introducing bisimplicial objects, we are in a position to characterize $\wedge$ by the commutativity of

$$\begin{array}{ccc} F_{m,p} & \xrightarrow{c} & F_n \\ {\scriptstyle\langle\forall,\forall\rangle}\downarrow & & \downarrow{\scriptstyle\forall} \\ F_{m-1,p-1} & \xrightarrow[c]{} & F_{n-1} \end{array}$$

and $\vee$ by the commutativity of

$$\begin{array}{ccc} F_{m,p} & \xrightarrow{c'} & F_n \\ {\scriptstyle\langle\exists,\exists\rangle}\downarrow & & \downarrow{\scriptstyle\exists} \\ F_{m-1,p-1} & \xrightarrow[c']{} & F_{n-1} \end{array}$$

$(n = \max(m, p))$,

$\neg$ is a simplicial morphism between $F_*^{\forall}$ and $F_*^{\exists}$:

$$\begin{array}{ccc} F_n & \xrightarrow{\neg} & F_n \\ \downarrow{\scriptstyle \forall} & & \downarrow{\scriptstyle \exists} \\ F_{n-1} & \xrightarrow[\neg]{} & F_{n-1} \end{array}$$

(commutative diagram).

Small problem with conjunction of formulas that share the same variables. For example, how to regiment a usual formula such as $P(v_0, v_1) \wedge Q(v_0, v_2)$ (where the same v_0 is at stake, which is important) into our framework?

$$
\begin{aligned}
P(v_0, v_1) \wedge Q(v_0, v_2) \quad &\equiv \quad P(v_0, v_1) \wedge Q(v_2, v_3) \wedge (v_0 = v_2) \\
&\equiv \quad (P(v_0, v_1) \wedge (v_0 = v_2)) \\
&\wedge \quad ((v_0 = v_2) \wedge (v_1 = v_1) \wedge Q(v_2, v_3))
\end{aligned}
$$

Particular case: $S_n(T)$, the set of all incomplete n-types over some model of a complete theory T, with $Q = \exists$.
We require furthermore that types have been regimented into mentioning only the v_i's as variables.

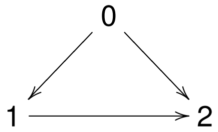
$$S_*(T) = \langle S_n(T), (\exists_i^n)_{0 \leq i \leq n}, (s_j^n)_{0 \leq j \leq n} \rangle_{n \in \mathbb{N}}$$

(with '$\exists_i$' being just a shorthand for $\exists v_i$) is a simplicial set.

Standard N-simplex: $\Delta^N = \mathrm{Hom}_\Delta(-, N)$.

$$\Delta^1 : \quad 0 \longrightarrow 1 \qquad \partial[v_0, v_1] = [v_1] - [v_0]$$

Δ^2 :



$$\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$

and so on.

n-chain = finite formal sum with positive or negative integers as coefficients.

$$|\Delta^N| = \{(t_0, \dots, t_N) \in \mathbb{R}^{N+1} : t_i \geq 0, \sum_{i=0}^{N} t_i = 1\}.$$

Hence, $|\Delta^1|$ is the segment $[0, 1]$, $|\Delta^2|$ the full triangle (the whole triangular surface), $|\Delta^3|$ the 3-dimensional analog in $\mathbb{R}^4$, and so forth.

The *realization* of a simplicial set X is the topological space $|X|$:

$$|X| = \varinjlim_{\Delta^n \to X} |\Delta^n|$$

Each $S_n(T)$ can be endowed with the usual topology generated by the the sets $[\varphi]$, where $[\varphi]$ = set of all n-types $p \in S_n(T)$ that contain φ. But $|S_*(T)|$ does not compare with any $S_n(T)$, because it comprises all the types whatsoever, whereas $S_n(T)$ is restricted to the n-types.

$S(T) = \coprod_{n \in \mathbb{N}} S_n(T)$ can be given the final topology associated to the family of inclusions $(\iota_n : S_n(T) \hookrightarrow S(T))_{n \in \mathbb{N}}$ (open subset of $S(T)$ = union of open subsets of some of the $S_n(T)$'s). Connections between the two topological spaces $|T_*|$ and $S(T)$?

Δ = category whose objects are all the finite ordinals, and whose morphisms are all order-preserving maps.
Simplicial set = functor $F : \Delta^o \to Set$
Simplicial object in $\mathcal{A}$ = functor $F : \Delta^o \to \mathcal{A}$.
Any simplicial object in the category of Abelian groups gives rise to an associated *chain complex*:

$$\cdots \qquad F_{n+1} \xrightarrow{d^{n+1}} F_n \xrightarrow{d^n} F_{n-1} \qquad \cdots$$

$$d^n = \sum_{i=0}^{n} d_i^n$$

$$d^n \circ d^{n+1} = 0$$

The homology groups of that complex can then be calculated.

$\langle F_n, \not\leftrightarrow, \bot\rangle$ is an Abelian group.
Problem: the d_i^Q's and the s_i's are not necessarily morphisms of Abelian groups. In particular,
$Qx\phi(v_0, \ldots, v_{i-1}, x, v_i, \ldots, v_{n-1}) \not\leftrightarrow$
$Qx\psi(v_0, \ldots, v_{i-1}, x, v_i, \ldots, v_{n-1})$
is not generally logically equivalent to
$Qx(\phi(v_0, \ldots, v_{i-1}, x, v_i, \ldots, v_{n-1}) \not\leftrightarrow$
$\psi(v_0, \ldots, v_{i-1}, x, v_i, \ldots, v_{n-1}))$.

First solution: Shift to sets of formulas:

$G_n = \wp(F_n)$
$\langle G_n, \Delta, \cap \rangle$ is a commutative group, where
$A \Delta B = (A - B) \cup (B - A)$
$\overline{d_i^Q} : G_n \to G_{n-1},\ \Gamma \mapsto \{d_i^Q(\varphi) : \varphi \in \Gamma\}$,
and in the same way $\overline{s_j} : G_n \to G_{n+1}$.

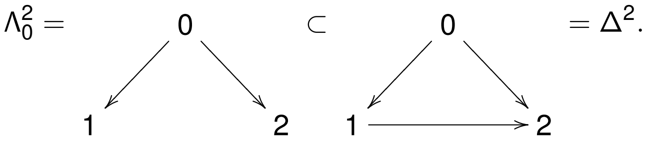
Second solution: Introduce the free Abelian group generated by the nondegenerate n-simplices. (A degenerate element is an element in the range of one of the s_j's.)

$C_n(F)$ = free Abelian group generated by the elements of F_n, with D_i in $C_n(F)$ taken to be the linear extensions of the face maps d_i^Q of F_*.

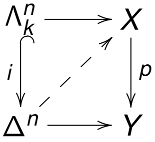
An element of the free Abelian group generated by F_n is a sequence $(\phi_1, \phi_2, \ldots, \phi_n, \overline{\phi_{n+1}}, \ldots, \overline{\phi_{n+p}})$, where $\overline{\phi_j}$ is the formal inverse of ϕ_j. That sequence can be seen in fact as a sequent $\phi_1, \phi_2, \ldots, \phi_n \vdash \phi_{n+1}, \ldots, \phi_{n+p}$.

Δ^n = standard simplicial set

k^{th} *horn* of Δ^n = subcomplex of Δ^n which is generated by all faces except the k^{th} one. For example:

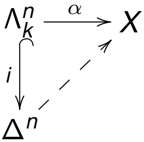


A *fibration* is a map $p : X \to Y$ of simplicial sets such that



In combinatorial terms:
If $x_0, \ldots, \widehat{x_k}, \ldots, x_n$ is an n-tuple of $n-1$-simplices of X such that 1) $d_i x_j = d_{j-1} x_i$ for any $i, j \neq k$ with $i < j$ and 2) there is an n-simplex y of Y such that $d_i y = p(x_i)$ for any $i \neq k$, then there is an n-simplex x of X such that $d_i x = x_i$ ($i \neq k$) and $p(x) = y$.

Fibrant simplicial set (or *Kan complex*) = simplicial set X such that the canonical map $X \longrightarrow *$ is a fibration:



For any Q, F^Q_* is fibrant. This is because only formal properties are at work. Semantics is required.

$$M_* = F_*^{\exists,M} = \langle F_n^{\exists,M} = \mathrm{Def}_{n+1}(M), (\exists_i^{n,M})_{0\leq i\leq n}, (s_j^{n,M})_{0\leq j\leq n}\rangle_{n\in\mathbb{N}}$$

with $\exists_i^{n,M} : A = \{\vec{a} \in M^{n+1} : M \vDash \varphi_A(v_0, \ldots, v_n)[\vec{a}]\} \mapsto \{\vec{a'} \in M^n : M \vDash \exists x\, \varphi_A(v_0, \ldots, v_{i-1}, x, v_i, \ldots, v_{n-1})[\vec{a'}]\}$ as face operators,
and $s_j^{n,M} : A \mapsto \{(\vec{x}, y) : \vec{x} \in A \text{ and } y = x_j\}$ as degeneracy operators.
M_* is a simplicial complex for any interpretation M of L.
$(F_{-1}^{\exists,M} = T(M))$

Let N be an extension of M. Then one can define the collection of maps

$$r_n : \mathrm{Def}_n(N) \to N_n^M,$$

where each r_n is the restriction mapping any definable subset $B \subset N^{n+1}$ to $r_n(B) = B \cap M^{n+1}$, and N_n^M is the set of all $\phi(M, N) := \{\vec{a} \in M^{n+1} : N \vDash \phi(v_0, \ldots, v_n)[\vec{a}]\}$ for $\phi \in F_n$. Furthermore, N_*^M can be endowed with a simplicial structure, with
$\exists_i^{N^M} : \{\vec{b} \in M^{n+1} : N \vDash \phi(\vec{v})[\vec{b}]\} \mapsto \{\vec{b'} \in M^n : N \vDash \exists_i(\phi)[\vec{b'}]\}$ for $\phi \in F_n$, and s_j defined accordingly.
Now, suppose that r_* is a simplicial map. Then it turns out N_*^M is in fact M_*, and that N is an elementary extension of M.

Theorem

A substructure M of a L-structure N is an elementary substructure of N iff the corresponding restriction $r_* : N_* \longrightarrow M_*$ *is a simplicial map.*

Corollary

The mapping $(-)_*$ *is a contravariant functor from the category of L-structures and elementary embeddings, to the category of simplicial sets and simplicial maps.*

Theorem

Let M be an elementary substructure of N. Then M_* *is a retract of* N_* *iff the domain* $|M|$ *of M is definable in N. (In that case,* M_* *and* N_* *are in particular homotopy equivalent.)*

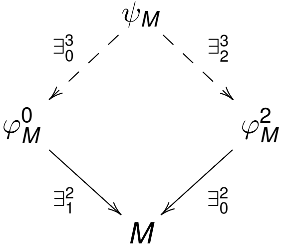
M being saturated amounts to infinitary conjunction being a simplicial map $S_*^{\exists}(M) \to M_*$:

$$\begin{array}{ccc} S_n(M) & \xrightarrow{\bigwedge} & \mathrm{Def}_n(M) \\ \exists \downarrow & & \downarrow \exists \\ S_{n-1}(M) & \xrightarrow[\bigwedge]{} & \mathrm{Def}_{n-1}(M) \end{array}$$

is a commutative diagram (where $S_n(M)$ here contains n-types that are supposed to be complete, or at least closed under conjunction).

Let A, B be two elements from $\mathrm{Def}_1(M)$ such that there is φ such that
$A = \{a \in M : M \vDash \exists v_1 \varphi(v_0, v_1)[a]\}$ and
$B = \{b \in M : M \vDash \exists v_0 \varphi(v_0, v_1)[b]\}$.
In that case, φ^M is said to be a *path* from A to B.
Being connected by a path = relation whose symmetric and transitive closure defines *path components* as equivalence classes.
$\pi_0(M)$ = set of all path components.

In the case of a *fibrant* simplicial set, the path relation is directly an equivalence relation.
But $F_*(M)$ is not generally fibrant. For $n = 2$, $F_*(M)$ is fibrant if, for any pair $(\varphi_M^0, \varphi_M^2)$ of members of $\mathrm{Def}_1(M)$



Let $p \in S_1(M)$ be a definable complete 1-type over some saturated model M.
To each formula $\varphi(v_0, v_1, \ldots, v_n)$ corresponds a formula $d_i^{p,n}(\varphi)(v_0, \ldots, v_{n-1})$ such that, for any tuple $(a_1, \ldots, a_n)$ of elements of M,

$$\varphi(a_1, \ldots, a_{i-1}, v_0, a_i, \ldots, a_n) \in p$$

$$\text{iff} \;\; M \models d_i^{p,n}(\varphi)(v_0, \ldots, v_{n-1})[\vec{a}]$$

$$d_i^{p,n}(\neg\varphi) = \neg d_i^{p,n}(\varphi)$$

and

$$d_i^{p,n}(\varphi \wedge \psi) = d_i^{p,n}(\varphi) \wedge d_i^{p,n}(\psi)$$

$$F_*^p = \langle\langle F_n, \not\leftrightarrow, \bot\rangle, (d_i^{p,n})_{0\leq i\leq n}, (s_j^n)_{0\leq j\leq n}\rangle_{n\in\mathbb{N}}$$

is a simplicial group.
Moore's theorem: the underlying simplicial set of a simplicial group is fibrant
Conclusion: F_*^p is fibrant.

$$\partial^p(\varphi) := \bigodot_{i=0}^{n} d_i^{p,n}(\varphi)$$

$\langle F_*^p, \partial^p\rangle$ is a chain complex ($\partial^2 = 0$).

Geometric flavor of a simplicial set such as F_* or M_*
Kouneiher & Balan, “Propositional manifolds and logical cohomology” (*Synthese* 125 (2000)):
Let $(P_i)_{i \in I}$ be a set of open Boolean algebras of propositions. A *propositional manifold P* can then be constructed by applications
$\varphi_i : I \to \wp(P_i),\ j \mapsto P_{i,j}$
and by continuous transition functions
$\phi_{i,j} : P_{i,j} \to P_{j,i}$ verifying $\phi_{i,i} = \text{id}$ and $\phi_{i,j} \circ \phi_{j,k} = \phi_{i,k}$.
The manifold P is given as a set as $\coprod_{i \in I} P_i / \sim$, where $x_i \sim x_j$ iff $x_j = \phi_{i,j}(x_i)$ for any $i \in P_i, j \in P_j$.

F_*^T (propositions are considered modulo T-interderivability) is a propositional manifold.
Suppose $\phi \in F_k$ and $\psi \in F_n$ are given, with $k \geq n$. Let's write then $\phi^{(i_1,\ldots,i_{k-n})}$ for
$\forall x_{i_1} \ldots \forall x_{i_{k-n}} (\phi[x_{i_1}/v_{i_1}, v_{i_1}/v_{i_1+1}, \ldots$
$\ldots v_{k-2}/v_{k-1}] \ldots [x_{i_{k-n}}/v_{i_{k-n}}, v_{i_{k-n}}/v_{i_{k-n}+1}, \ldots, v_{n-1}/v_n])$.

$\phi \equiv \psi$ iff there exists $(i_1, \ldots, i_{k-n}) \in \{0, \ldots, k-1\}^{k-n}$ such that $T \vdash \phi^{(i_1,\ldots,i_{k-n})} \to \psi$.
$[\phi] = \{\psi \in \bigcup_{n \in \mathbb{N}} F_n : \phi \equiv \psi\}$
$\varphi_n : \mathbb{N} \to \wp(F_n),\ k \mapsto \{\phi \in F_n : [\phi] \cap F_k \neq \emptyset\}$
$\phi_{n,k} : \{\phi \in F_n : [\phi] \cap F_k \neq \emptyset\} \to \{\psi \in F_k : [\psi] \cap F_n \neq \emptyset\},\ \phi \mapsto \psi \in [\phi] \cap F_k$.
We then get a manifold as it is defined just above.

Each assignment σ in *M* works as a *character*, that is as a function χ_σ over $F = \bigcup_{n\in\mathbb{N}} F_n$ with values in $\{0, 1\}$ such that:

- $\chi_\sigma(\neg\phi) = 1 - \chi_\sigma(\phi)$
- $\chi_\sigma(\phi \wedge \psi) = \chi_\sigma(\phi)\chi_\sigma(\psi)$
- $\chi_\sigma(\phi \dot{\vee} \psi) = \chi_\sigma(\phi) + \chi_\sigma(\psi)$ (where '$\dot{\vee}$' refers to exclusive disjunction)

Then: $\phi \vDash \psi$ implies that $\chi_\sigma(\phi) \leq \chi_\sigma(\psi)$ for any σ.

Alternatively:
$\widehat{M}$ = set of all partial assignments on M
Given $\sigma \in \widehat{M}$ and a fixed set of variables:

$$U_{\sigma,F} = \{\tau \in \widehat{M} : \sigma(x) = \tau(x) \text{ for any `}x\text{' in } F\}$$

as base open subsets. Putting:

$$U_{\sigma,F} \cap U_{\sigma',F'} := \begin{cases} U_{\sigma*\sigma',F\cup F'} & \text{if } \sigma \text{ and } \sigma' \text{ agree on } F \cap F' \\ \text{else } \emptyset, \end{cases}$$

where $\sigma * \sigma'$ is σ on F, σ' on F' and whatever on the remaining variables.
$\widehat{M}$ has been endowed with a Zariski-type topology.
Formulas then act on $\widehat{M}$ as differential forms:

$$\varphi(x_{i_0}, x_{i_1}, \ldots, x_{i_p}) : (\sigma_1, \ldots, \sigma_p) \mapsto \varphi[\sigma_j(v_j)/x_{i_j}] \in \{0, 1\}.$$

Each model of a category can be seen as a simplicial set.
Also: the category of all the models of a theory and elementary embeddings thereof, gives rise to a simplicial set (its “nerve”).

Application of “André homology”:

- the category $\underline{N}$ of all L-interpretations, for a first order language L;
- the category $\underline{M} = \mathrm{Mod}(T)$ of all models of T, for some formal theory T laid down in L;
- $F : \underline{M} \rightarrow Ab$, where Ab is the category of Abelian groups (for example, F associates to each model M of T the automorphism group of M, or F is a constant functor).

Under certain assumptions, one introduces, for any $N \in \mathrm{Ob}\underline{N}$:

$$s_1^0 : \sum\{F(M_1) : \ M_1 \xrightarrow{\alpha} M_0 \xrightarrow{\beta} N\} \hookrightarrow \sum\{F(M_1) : \ M_1 \xrightarrow{\beta\circ\alpha} N\}$$

and

$$s_1^1 : \sum\{F(M_1) : \ M_1 \xrightarrow{\alpha} M_0 \xrightarrow{\beta} N\} \xrightarrow{\sum F(\alpha)} \sum\{F(M_0) : \ M_0 \xrightarrow{\beta} N\}.$$

Then one puts: $d_1 = s_1^0 - s_1^1$, and likewise for d_2, d_3, and so forth. One then gets a simplicial complex $C_*^{\underline{M}}(N, F)$

$$\ldots \rightarrow \sum_{M_2 \rightarrow M_1 \rightarrow M_0 \rightarrow N} F(M_2) \xrightarrow{d_2} \sum_{M_1 \rightarrow M_0 \rightarrow N} F(M_1) \xrightarrow{d_1} \sum_{M_0 \rightarrow N} F(M_0) \xrightarrow{d_0} 0$$

If $N \in \mathrm{Ob}\underline{M}$, then

$$H_n(N, F) = \begin{cases} FN & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$$

Thus, homology here measures how far N is from being a model of T.

The realization of the nerve of $\underline{M}$, $X_T = |N(\mathrm{Mod}(T))|$, is a topological space that can be seen as a classifying space for T.

Simplicial sets = main example of a “model category”.

A *model category* is a category $\mathcal{C}$ equipped with three classes of morphisms called fibrations, cofibrations and weak equivalences, and having the following formal properties:

- $\mathcal{C}$ is closed under all finite limits and colimits.
- Suppose that
$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle f} & \\ & Z & \end{array}$$
commutes; if any two of f, g and h are weak equivalences, so is the third.
- The retract of a weak equivalence (resp. fibration, cofibration) is a weak equivalence (resp. fibration, cofibration).

- Suppose that we have a commutative solid arrow diagram

$$\begin{array}{ccc} U & \longrightarrow & X \\ {\scriptstyle i}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle p} \\ V & \longrightarrow & Y \end{array}$$

, where i is a cofibration and p is a fibration. Then, if either i or p is also a weak equivalence, then the dotted arrow exists, making the diagram commute.
- Any map f can be factored in two ways :
  - $f = p \circ i$ where p is a fibration and i both a cofibration and a weak equivalence, or
  - $f = q \circ j$, where q is both a fibration and a weak equivalence, and j a cofibration.

Case where $\mathcal{C}$ is (built up from) the category of models of a formal theory.
Problem of the retractibility conditions.
A way to go:
Every retract of an atomic compact structure is atomic compact.
There are some theories whose every model is atomic compact.

A L-structure M is said to be *atomic compact* if, for every set Φ of positive primitive formulas of L (*i.e.*, formulas of the form $\exists\vec{x}\,(\psi_1(\vec{x},\vec{a}) \wedge \ldots \wedge \psi_n(\vec{x},\vec{a}))$ with each ψ_i atomic) with parameters $\vec{a}$ in M and any number of free variables, if Φ is finitely realized in M then it is realized in M.

M is atomic compact iff for any elementary embedding $f : M \to N$ there is a homomorphism $g : N \to M$ such that $gf = \mathrm{id}_M$. This is in turn equivalent to the following condition: for any diagonal embedding $f : M \to M^U$ of M in some ultrapower of M, there is a homomorphism $g : N \to M$ such that $gf = \mathrm{id}_M$. In other words, M is a retract of any of its elementary extensions.

So here is a (sketchy) idea:
Consider the category of the models of some theory T with atomic compact projections (quasi-inverses of diagonal embeddings) as fibrations, embeddings as cofibrations, and isomorphisms as weak equivalences. More generally, the idea would be to compare the axioms of an abstract elementary class with the axioms defining a model category.

Gavrilovich :
Category of sets with morphisms $X \to Y$ iff
$\forall x \in X \, \exists y \in Y \, (x \subseteq y)$.
That category can be endowed with a model category structure, where homotopy amounts for two sets to differing by finitely many elements.

Category of the models of a theory = incomplete *model category*.

Conclusion:

Starting with syntactical considerations, we turned to models and then to categories of models (maybe even to model categories). This is a natural move after all, that can be seen as establishing bridges between Tarskian algebraic logic, Tarskian model theory and categorical model theory.