LiQ4 – Universalité et nécessité de la logique

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"Logical universalim:" Logical truths are both absolutely universal and absolutely necessary.

Main representatives of logical universalism: Frege and Russell.

In this talk I will focus on Frege and argue that Frege's universalism should not be conflated with a stiff absolutism: the defense of the universality of logic **is** compatible with the admission of variable *contexts* of discourse.

Main lines of my talk

I will:

- compare the respective ways in which Kant and Frege combine two different features commonly ascribed to logic
- trace back the label “logical universalism” to its root, and criticize it
- show how Frege’s perspective can countenance particular contexts of discourse
- sketch a formal framework to illutrate that point and compare Frege with Tarski
- show (*contra* Hodges and Demopoulos) that Frege *can*, as well as Tarski, make sense of non-logical constants as "indexicals."

Frege's "logical universalism"

Logic is a theory that precedes any other theory. The juridiction of logic knows no bound. This traditional characterization of logic is cashed out in terms of two distinct features:

- the **universality** of logic
- the **radicality** of logic

The universality of logic consists in its being about absolutely everything : only one type of entity variables ; a single unrestricted range of values for entity variables. **Nothing can escape logic.**

The radicality of logic corresponds to there being the one and the same logic that any reasoning must comply with. The principles of logic are laws that one cannot but presuppose them, even to challenge them. **Nobody can escape logic.**

The universality and the radicality of logic have been associated since at least Kant's *Logic*:

> *If [...] we put aside all cognition that we have to borrow from objects and merely reflect on the use just of the understanding, we discover those of its rules whih are necessary without qualification, for every purpose and without regard to any particular objects of thought, because without them we would not think at all. ("The Jäsche logic", Ak.* ix, *12)*

Logic is the "science of necessary laws of thought, without which no use of the understanding or of reason takes place at all" (Ak. ix, 13). **(Radicality)**
It is, as well, "a science a priori of the necessary laws of thought, not in regard to particular objects, however, but to all objects in general" (Ak. ix, 16). **(Universality)**

Frege himself gives in sometimes to a kind of confusion:

> *Here, we have only to try denying any one of them [one of the fundamental truths of arithmetic], and complete confusion ensues. Even to think at all seems no longer possible. The basis of arithmetic lies deeper, it seems than that of any of the empirical sciences, and even than that of geometry. The truths of arithmetic govern all that is numerable. This is the widest domain of all; for to it belongs not only the actual, not only the intuitable, but everything thinkable. Should not the laws of number, then, be connected very intimately with the laws of thought? (Foundations of Arithmetic, § 14)*

Asymmetry of Kant's and Frege's respective agendas

To Kant, radicality is a way of accounting for formal universality from within a nonformal conception of logic.
The domain ruled by formal logic can absolutely not be seen as more inclusive than the domain ruled by trancendental logic.
The generality of formal logic has to be reckoned through something else than its universality, namely: it states conditions of possibility of thought as such.

Frege's move goes the other way around: the universality of logic allows Frege to express the radicality of logic from within an anti-psychologistic conception of logic, without committing himself to some slippery notion of necessity.

To sum up: Both Kant and Frege bestow on logic two distinctive features: universality and radicality.

And both agree to consider those features as inseparable.

But Kant puts the stress on radicality, whereas *in Frege logical universality prevails over logical radicality*.

Logical universalism is more than the association of logical universality and logical radicality.
It is the sheer conflation of both, and the further claim that "one cannot consider one's language from the outside".

Although a label that has been pinned on to Frege, it is *not* epitomized by Frege himself.

Logical universalism is more of a reconstruction by authors who fought a family of philosophical stances represented by Frege, Russell, Wittgenstein and, to some extent, Carnap and even the young Tarski.

A tradition initiated by van Heijenoort put in the same "universalist" bag four independent claims:

- logic resorts to asolutely unrestricted (unspecified) variables
- logic refers to one single universe of discourse
- the language of logic is unescapable
- meta-theoretical studies, and semantics to begin with, are impracticable and, for that reason, banned.

Hintikka about the "ineffability of semantics:"

> *One of the main consequences of the universality of language (universality of the language) is that I cannot in my language speak of how its semantical relations to the world could be changed, at least not in a large scale. But such a systematic variation of the interpretation of a language is what the model theory for this language is all about. To speak of different models of a theory or a language in a logician's sense is ipso facto to speak of different systems of referential relations (interpretations) connecting language (or a part thereof) with the world. Hence all model theory is impossible according to the strict constructionist version of the universalist assumption. [Hintikka(1997)], p. 216)*

Otherwise put: there is but a single interpretation of language available because there is but a single universe of discourse because there is but a single world.

The universality of logic does not preclude one from considering local universes of discourse, in particular for metatheoretical purposes.

- *Principles* already, §430: Russell was well aware that an axiomatic system (Dedekind's arithmetic) may lend itself to various different interpretations.
- Russell's perspectivist reconstruction of the external world
- Philippe de Rouilhan ([de Rouilhan(2012)], p. 573-574): Russell, and Frege as well, did not, but *could have* introduced models of a theory within a universal logical system (a system based on the use of universal variables) and thus *could have* built up, within such a system, the equivalent of Tarskian model theory.

Several scholars (Tappenden, Heck, Landini) already showed that there was no objection of principle, nor in Frege neither in Russell, against "meta-theoretic" considerations.

I would like to pick up Frege's suggestion about geometry, and show **that Frege's upholding of the universality of logic does not preclude him from making sense of contextuality**.

Frege's work on the foundations of geometry, in the context of independence proofs in geometry, shows that the existence of an absolute, universal domain of quantification is compatible with the introduction of various systems of thoughts, which can be conceived of as reinterpretations of the language of mathematics.

Frege, “On the Foundations of Geometry” (1903):

> *[...] Euclidean geometry presents itself as a special case of a more inclusive system which allows for innumerable other special cases – innumerable geometries, if that word is still admissible. [...] If one wanted to use the word “point” in each of these geometries, it would become equivocal. To avoid this, we should have to add the name of the geometry, e.g. “point of the A-geometry,” “point of the B-geometry,” etc. Something similar will hold for the words “straight line” and “plane.” ([Frege(1971)], p. 37)*

Obviously, all special geometries are not subdomains of a single geometrical domain, but correspond to different theoretical contexts.

So Frege was not insensitive to the idea of context variation (= of re-interpretation of the language).

Variable reinterpretations as internal translations:

> *Imagine a vocabulary: not, however, one in which words of one language are opposed to corresponding ones of another, but where on both sides there stand words of the same language but having different senses. [...] We may say in general that words with the same grammatical function are to stand opposite one another. Each word occurring on the left has its determinate sense —at least we assume this— and likewise for each one occurring on the right. Now by means of this opposition the senses of the words on the left are also correlated with the senses of the words on the right. Let this correlation be one-to-one, so that on neither the left nor the right is the same thing expressed twice. We can now translate; not, however, from one language to another, whereby the same sense is retained; but into the very same language, whereby the sense is changed. ([Frege(1971)], p. 107-108)*

Definition of dependence:

> *Let us now consider whether a thought G is dependent upon a group of thoughts Ω. We can give a negative answer to this question if, according to our vocabulary, to the thoughts of group Ω there corresponds a group of true thoughts Ω', while to the thought G there corresponds a false thought G'. For if G were dependent upon Ω, then, since the thoughts of Ω' are true, G' would also have to be dependent upon Ω' and consequently G' would be true.*
> *With this we have an indication of the way in which it may be possible to prove independence of a real axiom from other [real] axioms. ([Frege(1971)], p. 110)*

[Tappenden(2000)], p. 274:

> *Crucially, for Frege, thoughts are evaluated as having the truth-values they actually have. No true thought is treated as false or projected into counterfactual circumstances in which it is false. (Axioms will be paired with other thoughts some of which may be actually false.)*

Important difference with Tarski: a Fregean vocabulary $e \to e'$ relates expressions e and e' belonging to the *same* logically regimented *interpreted* language L. It is not an arbitrary map from one domain (L-structure) to another.

Logical contextuality

Frege's perspective is neither in principle, *nor in fact* incompatible with the recognition of various contexts of discourse.

Admittedly, such admission, in Frege's writings, pertains to geometry, rather than to arithmetic and logic.

But the language of geometry is part of *the* language, and Frege's suggestion geometry shows that the language lends itself, as a whole, to partial re-interpretations.
This has nothing to do with the fact that geometry is not as general as arithmetic and logic.

Claim (made by Wilfrid Hodges and William Demopoulos): Frege is unable to account for non-logical constants of a formalized language, and for indexicals in natural language as well.

Hodges suggested that non-logical constants of a formal language can be seen as indexicals within the space of structures for the language.

For example, the symbol '$\cdot$' of the group operation, in the language L of group theory, is interpreted, in the *context* of each L-structure, by a specific operation in that structure: in any group G, "[. . .] the symbol '$\cdot$' automatically refers to the operation labelled '$\cdot$' in the group G" ([Hodges(1986)], p. 150).

Non-logical constants are given a reference by being applied to a particular structure.

To Frege and Peano, a rigourous language is a logically regimented fragment of natural language.
A sentence of a “Frege-Peano language” is interpreted from the outset, and thus is in itself true or false.

According to Demopoulos, Frege’s blindness to non-logical constants would stem from his blindness to indexicals, which itself would stem from his blindness to *contextuality*:

> *One of the features of natural language that simply has no analogue in a Frege-Peano language is the presence of indexicals. Non-logical constants, being like indexical expressions in the way they function, would for this reason alone make it difficult to express Huntington’s axiomatization within a Frege-Peano language [...] for Frege the sense of a designating expression determines its reference ‘absolutely’, i.e. independently of such contextual considerations as are relevant in the case of non-logical constants. ([Demopoulos(1994)], p. 216-218)*

I now would like to challenge this kind of assessment.

Let L be a fixed language, a sublanguage of “the” language L_0 (for instance, L is the Frege-Peano version of the language of group theory),
and let σ be the signature of L (in the case of group theory, $\sigma = \{e, \cdot, (-)^{-1}\}$).

A *Fregean σ-structure S* consists of:

- a set Ω of expressions, together with a “vocabulary” $f : \Omega \to \Omega_0$ such that $\sigma \subseteq \mathrm{Im}(f)$, where Ω_0 is the set of all expressions of the language L_0
- a set T of thoughts expressed with the resources provided by Ω.

Each σ-structure is a context for the use of the elements of σ.

$f(e)$ = expression (of L_0) that e replaces in L: **in L, e means what $f(e)$ usually means (= means in L_0)**.
In case $f(c) = c$, S adopts the intended interpretation of ‘c’.

The theory T are all the thoughts asserted to be true in the σ-structure.

A *morphism* $S = \langle \Omega, T \rangle \to S' = \langle \Omega', T' \rangle$ *of Fregean* σ*-structures* is a vocabulary $g : \Omega' \to \Omega$ such that:

- the diagram $$\begin{array}{ccc} \Omega & \xrightarrow{f} & \Omega_0 \\ {\scriptstyle g}\uparrow & \nearrow{\scriptstyle f'} & \\ \Omega' & & \end{array}$$ commutes;
- for each $t' \in T'$, $g(t')$ or $\neg g(t') \in T$.

Let $\mathcal{F}$ be the category of Fregean σ-structures thus defined.

A σ-structure S is *σ-congruent* with a σ-structure S', written $S \equiv_\sigma S'$, iff there is a morphism $S \to S'$ of σ-structures such that $\sigma \subseteq \mathrm{Fix}(g)$.
The relation of σ-congruence (or its symmetrical closure) is an equivalence relation.

A *context* is a σ-congruence class of σ-structures, for some implicit signature σ. It is a complete class of σ-structures sharing the same re-interpretation of all the members of σ.

All σ-structures of the same σ-congruence class as S can be regarded as alternative possible worlds in that context.

It is natural to consider, above each σ-structure S, the hierarchy H_S of all the objects and concepts to which all thoughts in T refer.

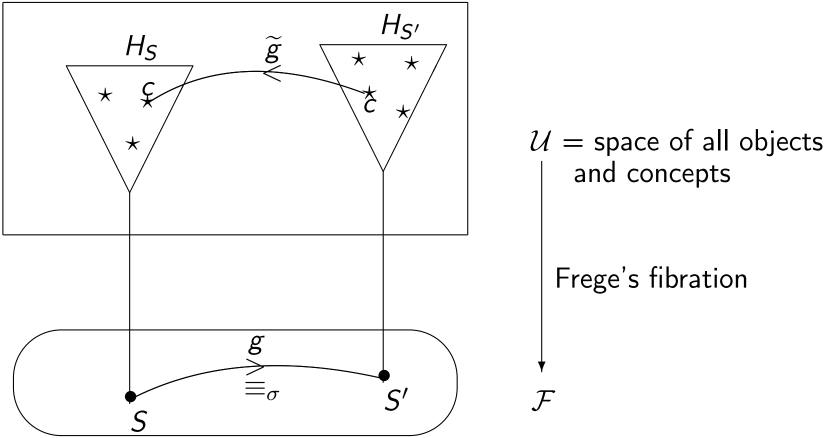
Any morphism $g : S \to S'$ between σ-structures induces a natural mapping $\widetilde{g} : H_{S'} \to H_S$ between the hierarchies based on those σ-structures.

In case the two structures S and S' are supposed to be σ-congruent, each non-logical constant c in the signature σ is preserved by $\widetilde{g}$.

Since the correspondence $S \mapsto H_S$ is "functorial," a *fibration* $p_F : \mathcal{U} \to \mathcal{F}$ can be defined over the category $\mathcal{F}$.

Fibrations are the category-theoretic generalization of surjective maps.
A *fibration* (or "fibered category") is a functor $\pi : E \to B$ such that, for any arrow $u : i \to j$ in B and for any object y above j (i.e., $j = \pi(y)$), there exists a "universal" arrow $\widehat{u} : x \to y$ in E above u ("above" meaning: $\pi(\widehat{u}) = u$).
$B =$ "base category", $E =$ "total space", $E_b := p^{-1}(b) =$ "fiber" above b.
Fibers generalize the set-theoretic notion of pre-image.

The fundamental extra-ingredient carried by a fibration, in comparison with pre-images, is the fact that the base category B is endowed with some structure (as embodied by its arrows), and that the existence of a fibration requires a systematic connection between the relations between any two objects in the base space, and the relations between the corresponding fibers in the total space. Base category = control space.



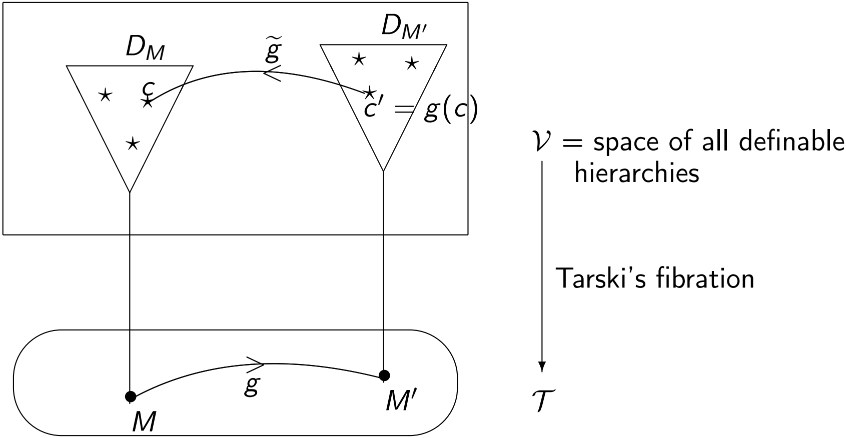
Frege's model theory

By way of comparison: Tarski's semantics.

Natural category attached to L = the category $\mathcal{T}$ whose objects are all L-structures, and whose arrows $g : M \to M'$ are all L-elementary embeddings $M \prec M'$
($M \vDash \phi[\vec{a}]$ if $M' \vDash \phi[f(\vec{a})]$ for all L-formulae ϕ and all elements $\vec{a}$ of M).

Natural system atached to each L-structure M: the hierarchy D_M of all definable subsets of M, i.e. all the subsets of $|M|^n$ equal to $\phi^M := \{(a_1, \dots, a_n) : M \vDash \phi[a_1, \dots a_n]\}$ for some L-formula $\phi(x-1, \dots, x_n)$.

Any elementary embedding $g : M \to M'$ induces a mapping $\widetilde{g} : \phi^{M'} \mapsto g^{-1}(\phi^{M'})$ from $D_{M'}$ to D_M.
So, again, the correspondence $M \mapsto D_M$ gives rise to a fibration $p_T : \mathcal{V} \to \mathcal{T}$ over $\mathcal{T}$ ("Tarski's fibration").



Tarski's model theory

Remark 1: In Tarski's semantics, *each* σ-structure M is a context of its own. But M is the unique possible world that can be considered in its own context (in context M).
Otherwise put, only diagonal or "primary" intentions can be represented.

Remark 2: Tarski's semantics should not be confused with Tarski's fibration.
In Tarski's semantics, one is always entirely free to shift from an L-structure M to another L-structure M', regardless of the connections between M and M'.
In Tarski's fibration, specific accessibility paths (elementary chains) are prescribed in the base category, and, accordingly, specific connections between fibers are distinguished.
Tarski's fibration : way of enriching Tarski's semantics.

Back to non-logical constants

In Frege's fibration as well as in Tarski's, a non-logical constant becomes nothing but a *section* of the fibration.

A *section* of p_F is a functor $s : \mathcal{F} \to \mathcal{U}$ such that $p_F \circ s = \mathrm{id}_{\mathcal{F}}$.

This amounts to the choice, above *each* object S of the base category $\mathcal{F}$, of a certain item $s(S)$ in the corresponding fiber $\mathcal{U}_S = H_S$, in such a way that, for any two objects S and S', $s(S) = \widetilde{g}(s(S'))$ for some morphism $g : S \to S'$ in $\mathcal{F}$.

So the choice made in each fiber is constrained insofar as all the choices thus made should comply with the conditions indicated by the morphisms of the base category.

The definition of a section of Tarski's fibration is analogous.

Of course, the view of non-logical constants as sections is almost tautological, since the signature σ is precisely built into those morphisms.
Still, it allows to picture and highlight non-logical constants as being indexicals.

The *role* (or the *character*) of a demonstrative, says Perry, is a "rule that takes us from each context of utterance to a certain object".
In the case of a non-logical constant (seen as an indexical), the rule at stake corresponds to the constraint set upon the *admissible* choices for a section of the fibration.

A non-logical constant, in the Fregean setting, as well as in the Tarskian one, consists of successive choices (along the morphisms of the base category) that have to make up a section.

Two conclusions:

- Understanding non-logical constants as indexicals is not beyond Frege's logical horizon.
- A non-logical constant, understood as a section, does not work in the same way in Frege and in Tarski. *A change of context, in Frege, is not a change of place, but an internal reorganization of language.*

By comparison, each non-logical constant, in Tarskian semantics, can be reinterpreted independently of the rest of the language. Accordingly, Tarski's semantics conflates contexts with domains and merges both into the mixed notion of "universe of discourse."

Believing that logical universality is incompatible with contextuality amounts to confusing a context variation with a domain variation.

But a context variation is not a domain variation, unless the principle of Tarskian semantics is presupposed.

It is actually possible to entertain a plurality of contexts from within a universal language.

And Frege's model theory makes it possible to analyze contextuality, and in a perhaps more satisfying way than Tarski's semantics.

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