Structured Variables

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Insolubia and solution by symbolic logic (1906a):

> *If this to be avoided, the range of significance must somehow be given with the variable; this can only be done by employing* **variables having some internal structures** *for such as are to be of some definite logical type other than individuals.*

Notes on Types (1906?):

> *I think it is quite impossible to set up mere formal rules which can be defended without going into the whole philosophy of the matter, and explaining that variables other than individuals* **have a recognizable structure***.*
>
> *What we can do, however, is to introduce the notion of variables which* **have a structure** ***relatively*** **to certain others***. This can be used, provided the other is presupposed.*

Insolubia and solution by symbolic logic (1906a):

> *Let p be a proposition, and a a constituent of p. Then '$p/a; b!q$' is to mean 'q results from p by substituting b for a whenever a occurs in p'. We then define [...] $p/a; b$ as 'the q which satisfies $p/a; b!q$'. [...] We call p/a the matrix of the substitution; it has no meaning by itself, since it stands for 'the result of replacing a in p by ...'.* **A matrix has all the formal properties of a *class***; *thus the members of p/a are the values of x for which $p/a; x$ is true, and so on. In order to insure that a statement about p/a holds of 'all classes', we have to state that it holds for 'all values of p and a', so that instead of one variable we have two. The notion of a class being a member of itself becomes meaningless; thought it is easy to construct a definition of what we mean when we say that a class is a member of a class of class.*

'$p/a; b!q$' means that q results from p by substituting b for a whenever a occurs in p

'$p/a, b; c, d!q$' means that q results from p by substituting c for a and d for b whenever a and b occur in p.

'$p/a; b$' means 'the result of replacing b for a whenever a occurs in p'.

Note: p, a, b, q, c, d have the same type.

$\exists F^{(o)} \forall x^o \, (F^{(o)} x^o \Leftrightarrow x^o = x^o)$ becomes:
$\exists p \exists a \forall x \, (p/a; x \Leftrightarrow x = x)$.

$\exists p \exists a \forall x \exists q \, (p/a; x!q \wedge \forall r \, (p/a; x!r \Rightarrow r = q) \wedge q \Leftrightarrow x = x))$.

$\exists G^{((o))} \forall F^{(o)} \, (G^{((o))} F^{(o)} \Leftrightarrow F^{(o)} = F^{(o)})$ becomes:
$\exists q \exists p \exists a \forall r \forall c \, (q/p, a; r, c \Leftrightarrow r/c = r/c)$.

$\exists G^{(o,o)} \forall x^o \forall y^o \, (G(x^o, y^o) \Leftrightarrow x^o = y^o)$ becomes:
$\exists q \exists p \exists a \forall r \forall c \, (q/p, a; r, c \Leftrightarrow r = c)$.

$\exists F^{(o)} \forall x^o \; (F^{(o)} x^o \Leftrightarrow x^o = x^o)$ becomes:
$\exists p \exists a \forall x \; (p/a; x \Leftrightarrow x = x)$.

$\exists p \exists a \forall x \exists q \; (p/a; x!q \wedge \forall r \; (p/a; x!r \Rightarrow r = q) \wedge q \Leftrightarrow x = x))$.

$\exists G^{((o))} \forall F^{(o)} \; (G^{((o))} F^{(o)} \Leftrightarrow F^{(o)} = F^{(o)})$ becomes:
$\exists q \exists p \exists a \forall r \forall c \; (\mathbf{q}/\mathbf{p}, \mathbf{a}; \mathbf{r}, \mathbf{c} \Leftrightarrow \underline{r/c = r/c})$.

$\exists G^{(o,o)} \forall x^o \forall y^o \; (G(x^o, y^o) \Leftrightarrow x^o = y^o)$
$\exists q \exists p \exists a \forall r \forall c \; (\mathbf{q}/\mathbf{p}, \mathbf{a}; \mathbf{r}, \mathbf{c} \Leftrightarrow \underline{r = c})$.

‘$p/a; (p/a)$’ would mean:

‘the result of replacing *a* with (the result of replacing *a* with . . . in *p*) in *p*’.

Landini 1998:

> *The substitutional theory is entirely type-free. There are no types of entities, and the language of the theory has individual variables as its only variables. Nevertheless [. . .] the substitutional theory proxies a type stratified calculus with predicate variables (in subject as well as in object position).*

The hierarchy of types is built directly into the logical grammar.

Insolubia and solution by symbolic logic (1906a):

> *The older symbolic logicians had a doctrine of the universe of discourse,* **setting***, as it were,* **bounds of decency, outside which no well-conducted variable would wander***. Thus when they asserted that (say) ϕx was always true, they only meant that it was always true so long as x was within the universe. Let us call the universe i. Then they really meant: " 'x is an i' implies ϕx."*

Insolubia and solution by symbolic logic, continued:

> **Thus a statement such as** ϕx **which is true under that hypothesis, can only be stated to be true under that hypothesis if the statement that the hypothesis implies** ϕx **can be made without any limitation on** x. *Any limitation on* x *is part of the whole which is really asserted; and as soon as the limitation is explicitly stated, the resulting implicational proposition remains true when the limitation is false. Thus a variable must be capable of all values. This argument may be fallacious, but I have never seen any attempt to refute it.*

Insolubia and solution by symbolic logic (1906a):

> *We thus find that we are brought back after all to variables with an unrestricted range.* **If this to be avoided, the range of significance must somehow be given with the variable; this can only be done by employing variables having some internal structure for such as are to be of some definite logical type other than individuals.**

Frege, *Grundlagen*, §66:

> *In the proposition 'the direction of a is identical with the direction of b' the direction of a plays the part of an object, and our definition affords us a means of recognizing this object as the same again, in case it should happen to crop up in some other guise, say as the direction of b.* **But this means does not provide for all cases.** *It will not, for instance, decide for us* **whether England is the same as the direction of the Earth's axis** *— if I may be forgiven an example which looks nonsensical. Naturally no one is going to confuse England with the direction of the Earth's axis; but that is no thanks to our definition of direction. That says nothing as to whether the proposition 'the direction of a is identical with q' should be affirmed or denied, except for the one case where q is given in the form of 'the direction of b.'*

Russell (1906b):

> *Thus, e.g. the question 'what is the number one?' will have no answer; the question which has an answer is 'what is the meaning of a statement in which the word one occurs?' And even this question only has an answer when the word occurs in a proper context [since] only certain kinds of statements can be significantly made about the number one.* **'The number one is bald' or 'the number one is fond of cream cheese' are, I maintain, not merely silly remarks, but totally devoid of meaning.**

Wittgenstein, *Dictations to Waismann*:

> *Let us imagine cubes, prisms and pyramids made of glass as being completely invisible in space. Only one surface of each prism, for example a square, and the base of each pyramid are supposed to be colored. We will then for instance only see squares in space. However* **we are unable to join together these plane figures arbitrarily because the bodies that are behind the surfaces prevent this**. *The law according to which the surfaces can be fitted together is determined by the invisible bodies whose surfaces are the squares. So I thought that the word had, as it were, a body of meaning behind it, and this body of meaning was to be described by the grammatical rules that hold true for the word. The grammatical rules would, as it were, be an unfolding of the nature of the meaning body.*

Letter to Whitehead 27 October 1904:

> *I send you herewith a somewhat rambling manuscript, ["On Functions"] containing a mixture of rhetoric and aspiration. In my own feelings, it embodies a distinct advance: I have begun to feel the Contradiction to be obvious and just what one might have expected; also the functionality (or the reverse) of this or that complex begins to seem obvious on inspection, except in very marginal cases.*

Floyd 2007:

> *[Some] conceptual inquiries have the peculiar feature that once an answer has been found, one can't wonder about whether or not the question is really settled. For part of what it is to give an answer is to settle on a proper understanding of what the original question was. To doubt the answer would then be to doubt one's understanding of the question. Where there was a riddle, there remains none — [. . .] a question one had no idea how to solve is no longer asked. In the words of the Tractatus, it vanishes as a problem.*

Williamson 2003:

> *The generality-relativist faces a problem of a kind that Wittgenstein sketches in the Preface to Tractatus-Logico-Philosophicus: 'in order to be able to set a limit to thought, we should have to find both sides of the limit thinkable (i.e. we should have to be able to think what cannot be thought)' (TLP 3). What, according to the generality-relativist, cannot be thought?*

What the generality-relativist says is that in a metalanguage L* for a language L:

(§1) It is impossible to quantify in L over everything.

Since (§1) is a sentence of L* and not of L, it is not self-defeating.

The generality-relativist wants to add:

(§2) It is impossible to quantify in L* over everything .

But of course, if the generality-relativist utters (§2) as a sentence of L*, the paradox recurs.

Notes on Types (1906?):

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- Hypothetical non logical ‘restriction’
- Logical restriction: structuration of the variable
- Non structural Tarskian logical restriction.

Restriction upon the range of values of a variable:

- Russell: type theoretic constraints built into the substitutional syntax;
- Tarski: external restriction of the domain of values.

From this it could seem that a possible value for a variable is either syntactically structured or structureless. Between these two extremes, it will be argued that a middle course can be steered.

Taking contexts seriously. Working with contexts that are not strictly syntactic (in opposition to Russell's substitutional theories), but that are not externally restricting “universes of discourse” either (in opposition to Tarskian semantics).

Natural framework: fibrations for type theories, where syntactic contexts are used to make up a canonical model. I will first set out an example of type theory, and then expound the idea of fibration.

$a : \sigma$.

$$\frac{\Gamma \vdash a_1 : \tau_1 \qquad \ldots \qquad \Gamma \vdash a_k : \tau_k}{\Gamma \vdash P(a_1, \ldots, a_k) : \mathrm{Prop}}$$

$\Gamma = \{x_1 : \sigma, \ldots, x_n : \sigma_n\}$.

Other rules for well-formed propositions (formation rules):

$$\frac{\Gamma \vdash \varphi : \mathsf{Prop} \qquad \Gamma \vdash \psi : \mathsf{Prop}}{\Gamma \vdash \varphi \wedge \psi : \mathsf{Prop}}$$

$$\frac{\Gamma \vdash \varphi : \mathsf{Prop} \qquad \Gamma \vdash \psi : \mathsf{Prop}}{\Gamma \vdash \varphi \to \psi : \mathsf{Prop}}$$

$$\frac{\Gamma, x : \sigma \vdash \varphi : \mathsf{Prop}}{\Gamma \vdash \forall x : \sigma . \varphi : \mathsf{Prop}}$$

A sequent $\Gamma \parallel \Theta \vdash \psi$ means that, in the context Γ, the proposition ψ follows from the set of propositions Θ.
Sequent calculus (transformation rules):

$$\frac{\Gamma \parallel \Theta \vdash \varphi \qquad \Gamma \parallel \Theta \vdash \psi}{\Gamma \parallel \Theta \vdash \varphi \wedge \psi}$$

$$\frac{\Gamma \parallel \Theta, \varphi \vdash \psi}{\Gamma \parallel \Theta \vdash \varphi \rightarrow \psi}$$

$$\frac{\Gamma, x : \sigma \parallel \Theta \vdash \varphi}{\Gamma \parallel \Theta \vdash \forall x : \sigma . \varphi} \; x \text{ not free in } \Theta$$

It is natural to rephrase the sequent calculus of the previous slide through a logical category of sequents *Lg* above a syntactic category of contexts *Ct*.

Syntactic category of contexts *Ct*:

- objects = all contexts $\Gamma = \{x_1 : \sigma_1, \ldots, x_n : \sigma_n\}$;
- an arrow $f : \Gamma \to \Gamma'$ between $\Gamma = \{x_1 : \sigma_1, \ldots, x_n : \sigma_n\}$ and $\Gamma' = \{x'_1 : \sigma'_1, \ldots, x'_m : \sigma'_m\}$ is any sequence $\langle f_1, \ldots, f_m \rangle$ of terms such that
$$\Gamma \vdash f_j(x_1, \ldots, x_n) : \sigma'_j$$
can be derived for any $1 \leq j \leq m$.

The "fiber" above an object Γ of *Ct* is the collection of propositions proved to be well-formed in that context.
Logical category of sequents *Lg*:

- objects = all couples $(\Gamma; \chi)$ of a context Γ and a proposition χ in that context: $\Gamma \vdash \chi : \mathrm{Prop}$;
- an arrow $\widehat{f} : (\Gamma; \chi) \to (\Gamma'; \chi')$ is an arrow $f : \Gamma \to \Gamma'$ in *Ct* such that

  $$\Gamma \parallel \chi \vdash \chi'[f_j/x'j]$$

  for all $1 \leq j \leq m$.

For example, an object of *Lg* is a well-formed proposition $\varphi(x)$ in context $x : \sigma$:

$$x : \sigma \vdash \varphi(x) : \mathrm{Prop}$$

(which does not mean that $\varphi(x)$ is true).

An arrow f in *Lg* from $(x : \sigma; \varphi(x))$ to $(y : \tau; \psi(y))$ is a term $f(x)$ such that:

$$x : \sigma \vdash f(x) : \tau$$

and

$$x : \sigma \parallel \varphi(x) \vdash \psi[f(x)/y] \quad .$$

There is an obvious projection $\pi_0 : Lg \to Ct$ mapping any proposition-in-context-Γ $(\Gamma; \chi)$ to Γ.

All propositions in context Γ make up a subcategory $(Lg)_\Gamma$ of Lg, called the *fiber above* Γ.

π_0 is a *fibration*.

For a fibration $\pi : E \to B$, $\pi^{-1}(b)$ = *fiber* E_b above $b \in B$.
Looking upwards, $\pi : E \to B$ = family of categories E_b indexed by B.

Example of set-indexed families of objects:
the base category is the category of all sets, and the fiber above any set I is the category Fam_I of all I-indexed families $(A_i)_{i \in I}$.
Then, given any family $(A_j)_{j \in J}$ above J and any arrow $u : I \to J$ there is an inclusion arrow $(A_{u(i)})_{i \in I} \to (A_j)_{j \in J}$, and $(A_{u(i)})_{i \in I}$ is the result of applying a substitution (reindexing) to $(A_j)_{j \in J}$.

Base category = control space.
Arrows in the base category become substitution operators in the total category.

The fibration $\pi_0 : Lg \to Ct$ is called a *syntactic fibration*.

In that fibration, a judgment such as $\Gamma \parallel \phi \vdash \psi$ becomes a logical relation in the fiber over the type context Γ. Hence, each such type context becomes an index for a logic describing what happens in this context.
Logic is about what happens in each fiber and also between fibers.

Idea: consider values whose structure would come from that of type contexts as captured by a syntactic fibration.

Tarskian semantics, clause for the interpretation of quantification in a first-order structure M w.r.t. some assignment σ in M:

$M \models \forall x \phi(x)[\sigma]$ iff, for any assignment σ' in M that differ with σ at most at x, $M \models \phi(x)[\sigma']$.

Variables of the language are replaced with variables of assignments.

Tarskian semantics in fibrational terms:

- Base category = collection of juxtaposed “universes of discourse,”
- Fiber above D = set of all assignations in D.

Base category S := category of all structures for a first-order language L, with all homomorphisms as arrows.
For any $M \in S$, $A_M := |M|^{\text{Var}}$ = set of all assignations in M.
Total category A := union of all A_M, $M \in S$.
Any arrow $f : M \to N$ in S gives rise to $\widehat{f} : \sigma \in A_M \to f \circ \sigma \in A_N$.

$A_M \mapsto M$ = Tarski's fibration $\pi_T : A \to S$.

So Tarski's semantics is amenable to fibrational semantics, as a kind of limit case.
Domains of the base space control are indeed integrated to the extent that maps relate them to each other, but those maps are only the ones that happen to exist in the background set-theoretic universe, and are not logical in nature.

Now, what about Russell's substitutional theory of classes?

Caveat: it is not about getting hold of structured variables, but rather about getting hold of structured values for a variable.
Indeed, Russell's substitutional theory does not explain what a variable is, but, on the contrary, presupposes the possibility of using variables. Still, Russell manages to assign to certain variables a specific range of possible values without restricting that range, by stating conditions built into the formalism that is used.
The environment of a syntactic fibration is the one that make it possible to bring to the fore values that would be internally structured as in Russell, but semantically interpreted as in Tarski.

Let p/a be a class, $q/p/a$ be a class of classes and $r/q/p/a$ be a class of classes of classes.
Saying that $p/a \in q/p/a$ becomes:

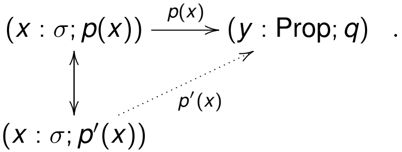
$$x : \sigma \parallel p(x) \vdash q(p(x)) \quad (1_q).$$

The derivability of (1_q) exactly amounts to the existence of $p(x)$'s being an arrow $(x : \sigma; p(x)) \to (y : \text{Prop}; q)$.

$p(x)$ = presentation of the class p/a.
Suppose that $p'(x)$ is a condition that is extensionally equivalent to $p(x)$:

$$x : \sigma \parallel p(x) \vdash p'(x) \ \text{ and } \ x : \sigma \parallel p'(x) \vdash p(x) \quad ,$$

which means that there are two arrows $(x : \sigma; p(x)) \to (x : \sigma; p'(x))$ and $(x : \sigma; p'(x)) \to (x : \sigma; p(x))$. As a consequence, any arrow $(x : \sigma; p(x)) \to (y : \text{Prop}; q)$ induces an arrow $(x : \sigma; p'(x)) \to (y : \text{Prop}; q)$:



$p(x)$ and $p'(x)$ are equivalent arrows, and $p(x)$ matters only as a representant of its equivalence class.

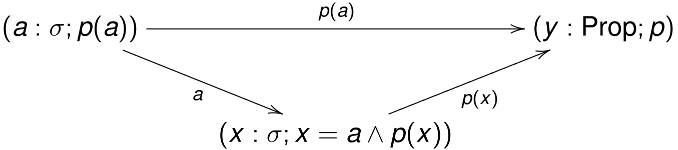
Class p/a:

$$a : \sigma \parallel p(a) \vdash p[p(a)/y] \ \ (1_p) \ \ \text{because } p[p(a)/y] = p,$$

$$a : \sigma \parallel p(a) \vdash (x = a \wedge p(x))[a/x] \ \ (2_p)$$

because $(x = a \wedge p(x))[a/x] = (a = a \wedge p(a))$,

$$x : \sigma \parallel (x = a \wedge p(x)) \vdash p[p(x)/y] \ \ (3_p) \ \ \text{because } p[p(x)/y] = p.$$



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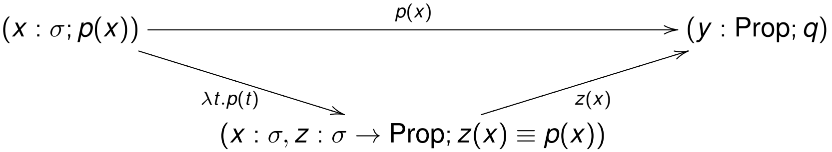
The arrow $p(a)$ represents p as a class p/a, and the arrow $p(x)$ represents p as the same class $p(x)/x$.

Class of classes $q/p/a$:

$$x : \sigma \parallel p(x) \vdash q(p(x)) \quad (1_q)$$

$$x : \sigma \parallel p(x) \vdash (\lambda t.p(t))(x) \equiv p(x) \quad (2_q)$$

$$x : \sigma, z : \sigma \to \mathrm{Prop} \parallel z(x) \equiv p(x) \vdash q(z(x)) \quad (3_q).$$



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The arrow $p(x)$ represents q as a class of classes $q/p/a$, whereas the arrow $z(x)$ represents q as a class q/p.

Class of classes of classes $r/q/p/a$:

$$x : \sigma \vdash q(p)(x) : \mathrm{Prop}$$

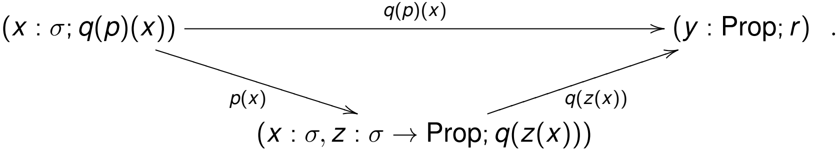
$$x : \sigma \parallel q(p)(x)) \vdash r(q(p)(x)) \ \ (1_r)$$

$$x : \sigma \vdash p : \sigma \to \mathrm{Prop}$$

$$x : \sigma \parallel q(p)(x) \vdash q(p(x)) \ \ (2_r)$$

$$x : \sigma, z : \sigma \to \mathrm{Prop} \vdash z(x) : \mathrm{Prop} \vdash q(z(x)) : \mathrm{Prop}$$

$$x : \sigma, z : \sigma \to \mathrm{Prop} \parallel q(z(x)) \vdash r(q(z(x))) \ \ (3_r).$$



The arrow $q(p)(x)$ represents r as a class of classes $r/q/p$, whereas the arrow $q(z(x))$ represents r as a class r/q.

$$
\begin{array}{ccc}
(x:\sigma; p(x)) & \xrightarrow{\quad p(x) \quad} & (y:\mathrm{Prop}; q) \\
& \searrow^{\lambda t.p(t)} \quad\quad \nearrow^{z(x)} & \\
& (x:\sigma, z:\sigma\to\mathrm{Prop}; z(x)\equiv p(x)) &
\end{array}
$$

Different arrows having the same target $(y : \mathrm{Prop}; q)$ = different ways of looking at q.
So a class such as $q/p/a$ is a bunch of arrows with different domains, but with the same target $(y : \mathrm{Prop}; q)$. All those arrows make up a category $Lg(q)$. One has the following composite fibration:

$$Lg(q) \xrightarrow{\mathrm{dom}} Lg \xrightarrow{\pi_0} Ct \quad .$$

Two arrows f, g in the same fiber relative to π_q correspond to two extensionally equivalent presentations of $q/p/a$. You pick out one of them at each time. This gives a “section” of π_q.
Such a section is the presentation of $q/p/a$ as a *structured value*.

Tarski's semantics: for a *fixed* interpretation structure M, each assignment gives a specific value in M to *each* variable of the language.
For a fixed variable 'x', an "x-generalized assignment" is a choice function $\alpha \in \prod_{M \in S} |M|$, that selects a member of the domain of each structure M for the language.
An x-generalized assignment is nothing else but a section of Tarski's fibration π_T.
Section as above, but with deductive contexts replaced with domains.

Different values given by an x-generalized assignment = sequence of "counterparts" of an original value as one moves from one domain to another.
But there is no connection between two successive values (the choice is each time entirely free).

Tarski's semantics = degenerate case.

Base category of Tarski's fibration = amorphous collection of domains, which does not work as a space control.

Categorical logic as a way to plunge Russell and Tarski in some kind of common framework.

Syntactic fibration = middle point between built-in type theory (syntactic contexts à la Russell) and set-theoretic domains (semantic domains à la Tarski).

Tarskian semantics: external constraint (a given domain).
Structured values semantics = structural type-theoretic constraint, in the semantic setting of fibrations.