A Fregean model theory

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Main claim

Frege's logical universalism is compatible with contextuality.

More specifically:

- The defense of the universality of logic is compatible with the admission of variable *contexts* of discourse.
- A properly Fregean notion of model can be devised.
- An understanding of non-logical constants is not beyond Frege's logical horizon.

Outline

I will:

- compare the respective ways in which Kant and Frege combine two different features commonly ascribed to logic;
- criticize the label "logical universalism;"
- show how Frege's perspective countenances the existence of several contexts of discourse;
- draw on a suggestion made by Frege in his discussion of Hilbert, and construct a Fregean notion of model;
- set out a formal framework to illutrate that point and compare Frege with Tarski;
- take up a suggestion from Wilfrid Hodges and William Demopoulos that non-logical constants of a formal language can be compared to indexicals, and show, *contra* Hodges and Demopoulos, that Frege can, as well as Tarski, make sense of non-logical constants.

Two features commonly ascribed to logic

Logic is a theory that precedes any other theory. The juridiction of logic knows no bound. This traditional characterization of logic is often cashed out in terms of two distinct features:

- the universality of logic
- the radicality of logic

The universality of logic consists in its being about absolutely everything: a single unrestricted range of values for entity variables. Nothing can escape logic.

The radicality of logic corresponds to there being the one and the same logic that any reasoning must presuppose and comply with. Nobody can escape logic.

Kant's *Logic*

The universality and the radicality of logic have been associated since at least Kant:

> *If [...] we put aside all cognition that we have to borrow from objects and merely reflect on the use just of the understanding, we discover those of its rules which are necessary without qualification, for every purpose and without regard to any particular objects of thought, because without them we would not think at all.*
>
> *("The Jäsche logic", Ak.* ix, *12)*

John Macfarlane, “Kant, Frege, and the Logic in Logicism” (2002)

Even though Kant and Frege obviously do not conceive of logic in the same way, they concur, according to Macfarlane in characterizing logic through its “normative generality.”

Logic provides one with the norms with which any thought whatsoever must comply as such, “if one is to think at all.”

However, MacFarlane’s analysis somewhat conceals the asymmetry of Kant’s and Frege’s respective agendas.

Asymmetry of Kant's and Frege's respective agendas

Kant – Formal logic is in a sense more general than transcendantal logic, because it does not presuppose the possibility of experience. Yet the domain ruled by formal logic cannot be seen as extending the domain ruled by trancendental logic (possible experience). The special status of formal logic has to be reckoned through something else than its universality.
To Kant, radicality is a way to account for the universality of formal logic from within a nonformalist conception of logic.

Frege – Frege's move goes the other way around.
To Frege, universality is a way to account for the radicality of logic from within an anti-psychologistic conception of logic, i.e., without committing oneself to any slippery notion of necessity.

To sum up

Both Kant and Frege bestow on logic two distinctive features: universality and radicality.

And both agree to associate these two features.

But Kant puts the stress on radicality, whereas in Frege logical universality prevails over logical radicality.

Frege's "logical universalism"

Logical universalism, a label that has been pinned on to Frege, designates:

- the conflation of logical universality and logical radicality
- the further claim that "one cannot consider one's language from the outside."

This logical universalism is not epitomized by Frege. Not any more, for that matter, by Russell, Wittgenstein, Carnap or even the young Tarski.

Logical universalism is an invention of its enemies.

The “no-metatheory” interpretation

A tradition upheld by van Heijenoort and Hintikka put in the same “universalist” bag four independent claims:

- logic resorts to absolutely unrestricted (unspecified) variables
- logic refers to one single universe of discourse
- the language of logic is a universal medium in which one can say everything.
- meta-theoretical studies, and semantics to begin with, are impracticable and, for that reason, banned.

This obviously does not square with the distinction that Frege emphasizes between the “expository language” (*Darlegungssprache*) and the “auxiliary language” (*Hilfessprache*) of his logical system. The latter is certainly not universal in the sense of the ordinary language, and the former eludes in some respects the rules set out in the formal system, as the concept ‘horse’ paradox shows.

The working hypothesis

Logical universality (“absolute generality”) is incompatible with ZFC-based model theory.

Logical universalism à la Hintikka is incompatible with any semantic metatheory.

But the defense of logical universality does not imply logical universalism.
Frege’s conception of logic is maybe incompatible with Tarskian model theory, but it should not be deemed *de jure* incompatible with any semantic metatheory as such.

There is no reason why it should be incompatible with a semantics which is not based on set-theoretic “universes of discourse”.

Frege, “On the Foundations of Geometry” (1903)

The universality of logic does not exclude the consideration of local universes of discourse, in particular for metatheoretical purposes.

Frege himself supplies an illustration of this point in his 1903 paper:

> *[. . .] Euclidean geometry presents itself as a special case of a more inclusive system which allows for innumerable other special cases — innumerable geometries, if that word is still admissible. [. . .] If one wanted to use the word “point” in each of these geometries, it would become equivocal. To avoid this, we should have to add the name of the geometry, e.g. “point of the A-geometry,” “point of the B-geometry,” etc. Something similar will hold for the words “straight line” and “plane.”*

Obviously, all special geometries are not subdomains of a single geometrical domain, but different specifications of the meaning of fundamental geometrical concepts, and thus correspond to different theoretical *contexts*.

Logical contextuality

So Frege was not insensitive to the idea of context variation.

Many scholars (Landini, Tappenden, Heck, Antonelli, Sullivan, Korhonen, Blanchette) already showed that there is no objection of principle in Frege against "meta-theoretic" considerations. Most of them refer to Frege's 1906 "On the Foundations of Geometry."

However, the focus of the reconsideration of Frege has remained mostly truth: Frege's metatheoretic samples pertain to semantics understood as the theory of reference and truth.
The now very substantial material adduced by the no-"no-metatheory" interpretation calls for a stronger claim:
Not only Frege's conception of logic countenances metatheoretic considerations, but Frege's suggestions in his 1906 essay hint at a genuine model-theoretic framework, although of course not of the Tarskian sort.

Frege's 1906 "Foundations of Geometry"

Frege introduces variable reinterpretations as internal translations called vocabularies:

> *Imagine a vocabulary: not, however, one in which words of one language are opposed to corresponding ones of another, but where on both sides there stand words of the same language but having different senses. [...] We may say in general that words with the same grammatical function are to stand opposite one another. Each word occurring on the left has its determinate sense —at least we assume this— and likewise for each one occurring on the right. Now by means of this opposition the senses of the words on the left are also correlated with the senses of the words on the right. Let this correlation be one-to-one, so that on neither the left nor the right is the same thing expressed twice. We can now translate; not, however, from one language to another, whereby the same sense is retained; but into the very same language, whereby the sense is changed.*

Tappenden, “Frege on axioms, indirect proofs, and independence arguments in geometry” (2000):

> *Crucially, for Frege, thoughts are evaluated as having the truth-values they actually have. No true thought is treated as false or projected into counterfactual circumstances in which it is false.*

Important difference with Tarski: A Fregean vocabulary $e \to e'$ relates expressions e and e' belonging to the same interpreted language.

Two objections

Admittedly, the recognition of various contexts of discourse, in Frege's writings, pertains to geometry, rather than to arithmetic and logic.

Still, nothing bars the extension of Frege's suggestion to a general framework, beyond the strict discussion of geometry (even though Frege does not deem geometry to be as general as arithmetic and logic).

Admittedly, Frege rejects any vocabulary which would change the sense of a word standing for a logical constant. Logical terms should be left invariant by any vocabulary.

But that is precisely what happens in modern semantics as well.

Two obstacles

Actually, Frege himself mentions two obstacles standing in the way of this strategy based on vocabularies:

> *We will find that this final basic law [the formality of logical laws according to which each object or first-level concept can be replaced, as far as logic is concerned, with any other] which I have attempted to elucidate by means of the above-mentioned vocabulary still needs more precise formulation, and that to give this will not be easy. Furthermore, it will have to be determined what counts as a logical inference and what is proper to logic.*

The first obstacle is the fact that no syntactic result is able to establish the kind of *absolute* logical dependencies that Frege intends to bring out.
The second obstacle is the problem of delineating the class of logical constants.

Answer to the first obstacle

P. Blanchette, "Frege on Formality and the 1906 Independence-Test" (2014): Each syntactic representation of logical relations between thoughts by a system of derivations between sentences corresponds only to a certain decomposition of logically complex notions, linked to a particular choice of primitives. Syntactic independence does not ensure logical independence.

Answer: Establishing facts about derivability w.r.t. a certain syntactic representation of logic is not a second-best option, even in Frege's view.

Answer to the second obstacle

Definition 1. Given two terms a, b with the same syntactic type, let v_a^b be the vocabulary (i.e., the reinterpretation of the language) induced by the replacement of 'a' with 'b'. For any sentence p of the language, $v_a^b(p)$ is the (possibly) new sentence generated from p by this vocabulary.

A thought expressed by a sentence p is valid (resp. contravalid) iff there exists some constituent u of p such that, for any term x of the same syntactic type as u, the thought expressed by $v_u^x(p)$ is true (resp. false).

A thought is (logically) determinate if it is either valid or contravalid.

Definition 2 (simplified). The class of logical terms is the largest class of terms, the composition of which can elicit but logically determined thoughts.

(This is an adaptation of the definition of logical terms given in Carnap's *Logical Syntax of Language*.)

Conclusion: One can do something about both obstacles standing in the way of Frege's suggestion.

Analysis of Frege's suggestion

Each "vocabulary" performs a context change, the replacement of certain semantic values with others. The meaning of a word is assigned as the new meaning of some other word.

In that framework, each context variation induces a systematic reinterpretation of language.
Indeed, an interpreted language is not a mere collection of words, but a system of meanings and references, and thus any change about one of them is always a simultaneous change of several interconnected meanings and references.

This is what complex names embody in the *Grundgesetze* (see §§29-31), were they object-names or function-names. Each complex name, considered as an autonomous syntactic unit, has a structure. And each context change has to reflect the dependencies that follow from the way complex names are formed.

Formalization

Any technical apparatus fit to represent Fregean context change should possess the following features:

- Each context appears as one item within some space of all possible (partial) reinterpretations of the language.
- To each context corresponds a certain system of interconnected semantic values, and each context change respects these connections.
- The relationships between any two contexts are mirrored by the relationships existing between the respective references of complex names as these names are interpreted in both contexts.

Fibrations

A certain technical framework naturally suggests itself: that of "fibrations."

Suppose that, to each object b of some category B, is assigned a category E_b, in such a way that any arrow $u : b \to b'$ in B gives rise to a functor $u^* : E_{b'} \to E_b$, called a "reindexing functor."

B is the *base category*, E_b is the *fiber* above b.

The double mapping $b \mapsto E_b$ and $u \mapsto u^*$ is a fibration over B.

(More exactly, the fibration is the corresponding functor $\pi : \coprod_b E_b \to B$ sending each object in E_b to b, and each arrow in E_b to 1_b, the identity on b.)
Fibrations generalize the set-theoretic notions of surjection and pre-image: $E_b = \pi^{-1}(b)$.

The fundamental extra-ingredient is the fact that the base category B is endowed with some structure (as embodied by its arrows), and that the existence of a fibration requires a systematic connection between the relations between any two objects in the base space (arrows u), and the relations between the corresponding fibers in the total space (reindexing functors u^*).

Base category = control space.

The framework of fibrations matching Frege's suggestion

- Each Fregean vocabulary is defined as a variant of the regular use of (certain) expressions, and as such belongs to some base variation space. This is the horizontal dimension of the picture.
- On top of each vocabulary stands a fiber: the complete referential system induced by the reinterpretation of certain expressions. This is the vertical dimension of the picture.
- Both dimensions are correlated, since each horizontal variation is matched by a correlative variation of what stands above each vocabulary. Hence a fundamentally two-dimensional picture.

Each vocabulary change is a partial shift of all semantic values inducing a transformation (reindexing functor) between structured systems (fibers).

This remains to be formalized.

Let L be a fixed language, a sublanguage of “the” language (let's call it L_0), and let σ be the signature of L.
For instance, if L is the language of group theory, then $\sigma = \{e, \cdot, (-)^{-1}\}$.

Objection - The very modern idea of signature is foreign to Frege, because the syntax of a formal language can never reveal the real primitives of a mathematical theory.
For instance, the real signature of arithmetic is actually empty, in Frege's view.

Answer - This objection has already been broached. Ultimately, it urges one to distinguish between what Frege would have been willing to do, and what one could do in order to pursue Frege's suggestion. What follows does not contend to reflect the former, but only the latter.
Once logical constants have been marked out, thanks precisely to the tool of vocabularies, nothing precludes considering a specific non-empty signature and applying to it a Fregean semantic apparatus.

Let Ω_0 be the set of all expressions of L_0.

A Fregean structure (or *F-structure*, for short) S consists of a set Ω of expressions such that $\sigma \subseteq \Omega$, together with a vocabulary $f : \Omega \to \Omega_0$.

The idea is the following: Each expression e in σ has, in the context of S, the meaning that $f(e)$ has normally in L_0.
As reinterpreted according to S, the meaning of an expression e is not the meaning of e, but the meaning of $f(e)$.
f = meaning shift.

In case $f(c) = c$, S simply sticks to the normal meaning of 'c'.

Each F-structure S can be understood as a context (in the broad sense) for the use of all the expressions in σ.

Example: shift from Euclidean to projective plane geometry

The signature σ of the language at stake contains the words 'plane', 'line' and 'incident'.

The F-structure $S_p = \langle \Omega_p, f_p \rangle$ representing projective geometry is specified by:

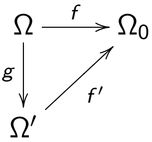
- f_p('plane') = 'surface of a sphere'
- f_p('line') = 'great circle (except the equator)'
- f_p('incident') = 'incident'.

When one uses the word 'plane' in the context of S_p, one actually means the surface of a sphere.

The thought expressed by 'Any two lines in the plane are incident' is normally false (because of the case of parallel lines).
The same sentence, as reinterpreted in S_p, expresses a thought that comes out true, namely the thought *Any two great circles on the sphere, both distinct from the equator, are incident*.

Reminder: An F-structure $S = \langle \Omega, f \rangle$ consists of a set $\Omega \supseteq \sigma$ of expressions together with a vocabulary $f : \Omega \to \Omega_0$.

A morphism $S = \langle \Omega, f \rangle \to S' = \langle \Omega', f' \rangle$ of F-structures is simply a vocabulary $g : \Omega \to \Omega'$ such that $f' \circ g = f$.



Let $\mathcal{F}$ be the category of all F-structures.

Two F-structures S and S' are σ-congruent iff there is a morphism $g : S \to S'$ such that $\sigma \subseteq \mathrm{Fix}(g)$ —which means that S and S' reinterpret all the expressions of σ in the same way.

The relation of σ-congruence (or its symmetrical closure) is an equivalence relation.

A context (in the narrow sense) is a σ-congruence class of F-structures, for some implicit signature σ. It is a complete class of F-structures sharing the same reinterpretation of all the members of σ.

All F-structures of the same context as S can be regarded as alternative possible worlds in that context.

It is natural to consider, above each F-structure S, the system R_S mentioning the reference of all simple and complex proper names, as well as the value-ranges of all function-names (of all levels), as these names are reinterpreted by S, together with all the dependencies between them.

This system R_S can easily be given the structure of a category:

- The objects of R_S are all referential pairs $\langle n, r \rangle$, where either 'n' is a proper name and r its S-reference, or 'n' is a function-name and r its S-value-range.
- The arrows of R_S are determined by the following rule: There is an arrow $\langle n_1, r_1 \rangle \to \langle n_2, r_2 \rangle$ each time 'n_1' forms part of 'n_2' or coincides with it. (see *Grundgesetze*, §26).

Any morphism $g : S \to S'$ in $\mathcal{F}$ induces a natural mapping $g^* : R_{S'} \to R_S$ that can be defined in such a way that, for instance: Given a complex name n

$$g^*(\langle n, r' \rangle) = \langle n, r \rangle$$

where r' is the actual reference of $f'(n)$, and r is the actual reference of $(f'g)(n) = f(n)$.
So for instance:

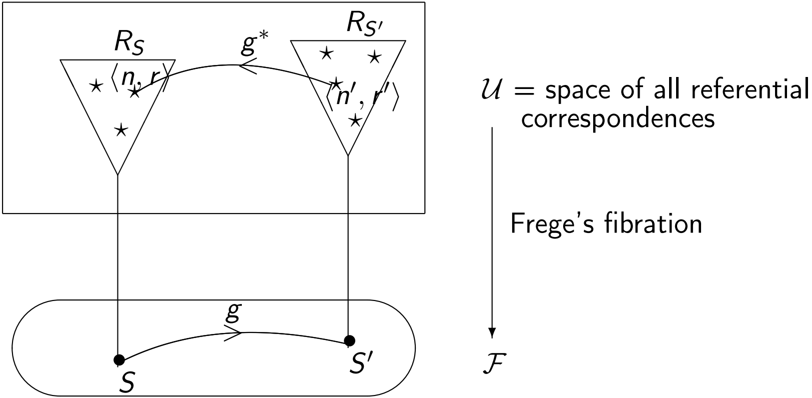
$$g^*(\langle \text{`plane'}, \text{plane} \rangle) = \langle \text{`plane'}, \text{sphere} \rangle.$$

In case S and S' are σ-congruent, the interpretation of each non-logical constant c in σ is preserved by g^*.

The correspondence $S \mapsto R_S$ is "functorial," hence one gets a fibration over $\mathcal{F}$, let's call it Frege's fibration.

A vocabulary shift (in Frege's sense) thus becomes a reindexing functor g^* in Frege's fibration.

Frege's fibration



Comparison with Tarski's semantics

A natural category attached to a first-order language L is the category $\mathcal{T}$ whose objects are all T-structures for L, and whose arrows $g : M \to M'$ are all L-elementary embeddings $M \prec M'$.

A natural system atached to each T-structure M is the hierarchy D_M of all definable subsets of M, i.e., all the subsets of $|M|^n$ of the form $\phi^M := \{(a_1, \ldots, a_n) : M \vDash \phi[a_1, \ldots a_n]\}$ for some L-formula $\phi(x_1, \ldots, x_n)$.
Each individual constant c is represented in M by the definable subset $(x = c)^M = \{c^M\}$.

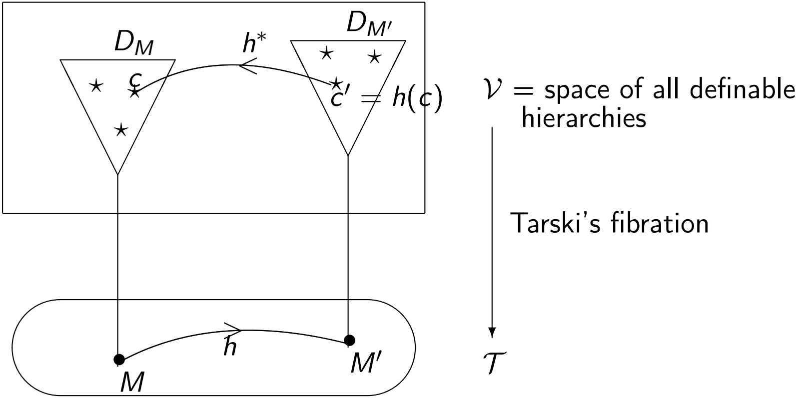
This hierarchy D_M can naturally be turned into a category:

- The objects of D_M are all definable subsets of M.
- The arrows of D_M are all definable maps between them.

Any elementary embedding $h : M \to M'$ induces a mapping $h^* : \phi^{M'} \mapsto h^{-1}(\phi^{M'})$ from $D_{M'}$ to D_M.

So, again, the correspondence $M \mapsto D_M$ allows one to define a fibration over $\mathcal{T}$, Tarski's fibration.

Tarski's fibration



Tarski's semantics, Tarski's fibration

Tarski's semantics should not be confused with Tarski's fibration. Tarski's semantics corresponds to the limit case of Tarski's fibration where no constraint is set upon the shift from one T-structure to another.

In Tarski's semantics, one is always entirely free to shift from a T-structure M to another T-structure M', regardless of the connections between M and M'.

In Tarski's fibration, on the contrary, specific accessibility paths (elementary chains) are prescribed in the base category, and, accordingly, specific connections between fibers are distinguished.

This is what Frege's fibration puts to the fore as well: Each particular signature σ induces a collection of contexts, which correspond to admissible paths from one F-structure to another.

Non-logical constants as indexicals

Wilfrid Hodges suggested that non-logical constants of a formal language can be seen as indexicals within the space of structures for the language.

For example, the symbol '$\cdot$' of the group operation, in the language L of group theory, is interpreted, in the *context* of each L-structure, by a specific operation in that structure: In any group G, "[...] the symbol '$\cdot$' automatically refers to the operation labelled '$\cdot$' in the group G."

Non-logical constants are given a reference by being applied to a particular structure, exactly as the word 'yesterday' can be given an actual reference by being used in a particular temporal context.

Frege's alleged blindness to contextuality

A claim made by Hodges and William Demopoulos is that Frege is unable to account for indexicals (in natural language) and for non-logical constants (of a formalized language).

According to Demopoulos, Frege's blindness to non-logical constants stems from his blindness to indexicals, which itself stems from his conception of the reference of designators as being determined in an "absolute" way.

The framework introduced above helps to challenge this kind of assessment.

Fregean non-logical constants

In Frege's fibration as well as in Tarski's, a non-logical constant becomes nothing but a section of the fibration.

A section of Frege's fibration amounts to the choice, above each F-structure S, of a certain item $s(S)$ in the corresponding fiber R_S, in such a way that, for any two F-structures S and S',

$$s(S) = g^*(s(S'))$$

for some morphism $g : S \to S'$.

The choice made in each fiber is constrained insofar as all the choices thus made should comply with the conditions indicated by the morphisms of the base category.

The definition of a section of Tarski's fibration is analogous, with the constraints indicated by elementary embeddings instead.

A section of Frege's fibration consists of a collection of items (referential pairs), one for each F-structure, so that any two σ-congruent F-structures share the same item as soon as this item reinterprets an expression in σ.

Similarly, a section of Tarski's fibration consists of a collection of items (definable subsets), one for each T-structure, so that any two T-structures with a morphism between them have matching items as soon as these items interpret the same expression in the signature.

This view of non-logical constants as sections is almost tautological, since the signature σ is precisely built into those morphisms.

Still, it allows to understand non-logical constants as indexicals in Tarski and in Frege as well.

The role or character (Kaplan) of a demonstrative, says John Perry, is a "rule that takes us from each context of utterance to a certain object."

In the case of a non-logical constant (seen as an indexical), the rule corresponds to the constraint set upon the admissible choices which add up to build a section of the fibration.

This stands in sharp contrast with what happens in Tarski's *semantics*, where the respective interpretations c^M and $c^{M'}$ of the same constant (or the respective interpretations φ^M and $\varphi^{M'}$ of the same formula) in two different structures M and M' can be considered independently of each other.

Applications of Tarski's fibration

- Dependencies between variables (J. van Benthem)
- Semantic relationalism (K. Fine)
  If two variables x and y (supposed to range the same domain of individuals) have the same semantic role when considered in two different expressions, such as $x > 0$ and $y > 0$, how to explain that they cannot have the same semantic role when they occur in the same expression, such as $x > y$?
  Fine speaks of a "cross-contextual difference" between the pair x, y and the pair x, x.

Fine 2007:

> *We must [. . .] recognize that there may be irreducible semantic relationships, ones not reducible to the intrinsic semantic features of the expressions between which they hold. [. . .] The picture of an assemblage of semantic snapshots must be supplemented by a picture in which these snapshots are connected, one to the other, by semantic threads.*

Two conclusions

1. Understanding non-logical constants as indexicals is not beyond Frege's horizon.
2. A non-logical constant, understood as a section, does not work in the same way in Frege and in Tarski.

Let me elaborate a bit on this second point.

A change of context, in Frege, is not a change of place, but an internal reorganization of the language.

In contrast, each non-logical constant, in Tarskian *semantics* (as opposed to Tarski's *fibration*), can be reinterpreted independently of the rest of the language.

Accordingly, Tarski's *semantics* conflates contexts with domains and merges both into the mixed notion of "universe of discourse."

This conflation is exactly the target of Frege's criticism of the invocation of "circumstances" of truth in his 1906 "Foundations":

> *A proposition that holds only under certain circumstances is not a real proposition. [...] If we suppose that a proposition can hold under certain circumstances but not under others, then we allow ourselves to be led by the nose by self-induced inexactitudes of expression.*

To Frege, the confusion of pseudo-propositions (formal sentences) with real (interpreted) propositions, and the confusion of contexts with circumstances are two facets of the same fundamental confusion.

A change of context is irreducible to a change of domain and to a change of circumstances as well.

To put it another way: In the context of consistency and independence proofs, Hilbert put forward the general concept of reinterpretation of a formal theory.

Tarski's semantics actually constitutes only a particular implementation of Hilbert's concept of reinterpretation.

The Fregean avenue based on the notion of vocabulary is precisely another implementation, which the apparatus of fibrations is geared to bring out and to develop.

CONCLUSION

1. Logical universality is perfectly compatible with contextuality. Believing that logical universality excludes contextuality amounts to confusing a context variation with a domain variation, by presupposing the principle of Tarskian semantics.
2. It is not true that Frege could make sense neither of non-logical constants nor of the idea of context. It is the other way around: Frege's 1906 suggestion makes it possible to analyze contextuality in a more satisfying way than Tarski's semantics does.
3. The apparatus of fibrations allows one to free interpretation of Frege from a technical artefact, and besides to enrich Tarski's semantics into Tarski's fibration.
4. I am not sure I have been entirely faithful to Frege. And I am not sure either that one cannot do without fibrations. However, I do believe that history of logic and mathematical techniques have to be combined, so as to unfreeze logical practice.