Mathematical settings

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Two goals:

- First, I would like to show that combinatorics provides a good example of what a *mis*understanding in mathematics can be, and may thus cast some light on both mathematical understanding and mathematical objectivity.
- This point comes close to the second goal of my talk, which is to reconsider the "identity problem" faced by the structuralist interpretation of mathematics.

Ante rem structuralism

According to Stewart Shapiro's *ante rem* structuralism, mathematics studies structures, conceived of as certain configurations of pure relata.

In that view, the constituents of a structure do not have any individuality, since each of them is entirely characterized by the bundle of relations which connect it to all the other constituents.

In other words, a mathematical object is but a "place" within a structure. "The number 2, for example, is no more and no less than the second position in the natural-number structure."

A given collection of individual objects, with relations between them, which exemplifies a certain structure, is what Shapiro calls a system.

A structure is "the abstract form of a system," a system is an instance of a structure.

However, a structure is not the mere result of its abstraction from the diverse systems that instantiate it.

Indeed, *ante rem* structuralism, as opposed to *in re* structuralism, takes the places of a structure to be full-fledged objects in their own right, as opposed to mere "offices."

As a consequence, the places-as-objects of a given structure make up a system which instantiates itself as a structure.

The identity problem

In the perspective of *ante rem* structuralism, a mathematical object, as a place in a structure, cannot be individuated beyond its structural properties within that structure.

This precipitates the "identity problem," as it has been brought up by Jukka Keränen.

This is the problem of explaining how two distinct objects of the same structure can have exactly the same structural properties and yet be distinct.

There are structures in which two distinct places are still structurally indistinguishable (despite being distinct).

Very elementary examples of that situation are the points of the geometric plane, or the conjugate complex numbers (i.e., $a + ib$ and $a - ib$).

In fact, any structure with a nontrivial automorphism is an instance of the identity problem.

How to conceive of two distinct yet structurally indiscernible places of the same structure?

Shapiro points out that it is always possible to differentiate two given distinct objects. To be specific, the pair $\langle i, -i \rangle$ satisfies the formula $x + y = 0$, whereas $\langle i, i \rangle$. does not. This suffices to vindicate the fact that i and $-i$ are distinct.

In Keränen's view, however, any theory of a certain kind of objects has to provide one with a general individuation criterion for any object. those objects.

Keränen: "[According to *ante rem* structuralism], whenever two distinct elements in a system have the same intra-systemic relational properties, [. . .] they occupy the same place of the corresponding structure."

But this assumption, according to Shapiro, is wrong: The non-coincidence of two places in a structure is shown by the non-coincidence of the two items that occupy those places in the system that is the structure.

The distinctness of two distinct, yet otherwise indistinguishable constituents of a structure constitutes a brute fact.

Leitgeb & Ladyman's example, taken from graph theory:

As Shapiro puts it, "we know what identity is."

Combinatorics

There is a privileged domain in mathematics which calls for its consideration —a domain where permutations and symmetries are brought to the fore, namely combinatorics.

Studying the very basic framework of combinatorics should give one the insight needed to understand how automorphisms and symmetries work, and thus how to reconsider the Keränen *vs* Shapiro controversy.

This will allow us to develop the second thread of this talk: misunderstanding in mathematics.

Suppose that three items a, b, c are given, and let's consider the permutation that swaps a and b while leaving c unchanged. This is a very elementary mathematical scenario. But is it so simple a thing to grasp?

The set composed of a, b and c is not altered by the permutation under consideration: This permutation is, so to speak, a mere fancy of the mind. Yet we cannot help but picture a, b and c spatially, assigning to those objects different respective positions.
Doing that, we really could have put originally a at the position that is actually occupied by b, and put originally b at a's actual position. (Or: We could have named a, 'b' —and vice versa.)
This would have changed nothing.

So a permutation refers to a setting which is itself defined only up to any arbitrary permutation of the original positions (or the names) of a, b and c.

Otherwise put:

The representation of a permutation involves arbitrary choices which make it appear as an invariant under typographical permutations.

So, to really understand how to represent a permutation, one ought to have some prior understanding of some permutations.

There is no vicious circle here, because the typographical permutations (of the labellings of the items on which the permutation under consideration acts) or the spatial permutations (of the initial positions of the items, as one pictures them) are not permutations in the proper (mathematical) sense.

Still things turn out to be trickier than one could expect them to be.

A permutation is a one-to-one mapping of a set to itself. One usually writes a permutation by using schematic letters. For instance, the permutation on a three-object set which exchanges the two first ones and leaves the third one untouched is usually written:

$$\begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}.$$

It is obvious that the choice of $\{a, b, c\}$ instead, say, $\{\alpha, \beta, \gamma\}$ or $\{a_1, a_2, a_3\}$, is completely immaterial.

This does not mean, however, that one is considering the set $\{a, b, c\}$ up to a permutation "replacing" a with α, b with β, and c with γ. Indeed, keeping track of permutations precisely presupposes that letters have been settled once and for all.

The letters a, b and c are variable parameters to the extent that they are arbitrary, but —of course— they are not variable in the sense of the variation that they make it possible to represent.

Threefold arbitrariness attached to any representation of a permutation:

- the arbitrariness of the underlying set:

$$\begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix} \equiv \begin{pmatrix} \alpha & \beta & \gamma \\ \beta & \alpha & \gamma \end{pmatrix}.$$

- the arbitrariness of the original arrangement of the members:

$$\begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix} = \begin{pmatrix} b & a & c \\ a & b & c \end{pmatrix}.$$

- the arbitrariness of the labelling of the members of that set:

$$\begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix} \sim \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix}.$$

The two permutations are said to be *conjugate*.

I am mainly interested in the latter —the arbitrariness of the labelling:

$$\begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix} \sim \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix}.$$

It looks like what the first permutation does with b is what the other does with c, and vice versa. It is as if both permutations were doing exactly the same thing, except that the labels of b and c have been swapped. And well, after all the object called 'b' could have been called 'c', and conversely.

In reality, it makes absolutely no sense to mark off the objects from their names. We do not have access to a otherwise than by its name. The very distinction between objects and names is confused.

So names do not matter.

Strictly speaking, however, the group of all permutations on a set X with n elements, written $\mathfrak{S}_n$, is defined as the group of all permutations on the particular set $\{1, 2, \dots, n\}$.

The identification is natural, since the group of all permutations $\mathfrak{S}_X$ on *any* n-element set X is isomorphic to $\mathfrak{S}_n$.

The isomorphism is not canonical. And selecting one precisely amounts to selecting a certain numbering of X.

But in the case $X = \{1, 2, 3, \dots, n\}$, the numbering is trivial, i.e., amounts to taking each number as its own numeral: $1 = 1, 2 = 2, \dots, n = n$.

So, in the usual representation of a permutation, the labels are the numerals directly corresponding to the numbers. The numbers act as their own numerals.

But, then, what happens if the numbers are confused with numerals?

It is not un-understandable to understand numerals 1, 2, 3, . . . as indices of their respective canonical ranks, rather than the underlying individual objects that these numerals are supposed to stand for throughout the permutation.

Let π the following permutation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}.$$

Here 1, 2, 3 et 4 are not ranks, but individual objects, whose orbits are considered as π is iterated:

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 2 & 1 \\ 1 & 2 & 3 & 4 \end{array} \quad .$$

Now let's imagine the following heterodox interpretation, let's call it Dummy's interpretation.

Dummy understands $1, 2, 3, 4$ as ranks rather than as objects, and accordingly carries out the iteration of π in this way:

$$R_1 \quad = \quad 1 \;\; 2 \;\; 3 \;\; 4 \quad = \quad 1.1 \;\; 2.1 \;\; 3.1 \;\; 4.1$$

$$R_2 \quad = \quad 4 \;\; 3 \;\; 1 \;\; 2 \quad = \quad 1.2 \;\; 2.2 \;\; 3.2 \;\; 4.2 \quad .$$

The second row R_2 is understood as reindexing the ranks:
4 becomes the "new" 1, so to speak, that is, the "new" number 1 position. Similarly 3 becomes the "new" 2, and so on.
In this perspective, 1 is not only 1, but first and foremost the index of the first position in the current row, and the permutation is viewed as a perturbation of the ranks in reference to $\{1, 2, \ldots, n\}$.

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}$$

According to Dummy's reading, the calculation of the third row R_3 for π goes that way:

> "4 *becomes* 2, *but since* 2 *now is* 3 *(because* $2.2 = 3$*),* 4 *finally lands on* 3. *So* 3 *will be the first numeral on* R_3*."*

The basic principle of that heterodox interpretation is that the 4 on the first row R_1 becomes 2 on the second row R_2, but 2 as understood precisely according to the new code set by R_2, namely as $2.2 = 3$. In the same way, 3 becomes 1 with $1.2 = 4$.

One finally gets:

$$R_3 \quad = \quad 3 \;\; 4 \;\; 2 \;\; 1 \quad = \quad 1.3 \;\; 2.3 \;\; 3.3 \;\; 4.3$$

To sum up, the correct iteration of π of course is

$$\begin{array}{lcccc} R_1 & 1 & 2 & 3 & 4 \\ R_2 & 4 & 3 & 1 & 2 \\ R_3 & 2 & 1 & 4 & 3 \end{array},$$

but in Dummy's interpretation, the iteration of π becomes

$$\begin{array}{lcccc} R_1 & 1 & 2 & 3 & 4 \\ R_2 & 4 & 3 & 1 & 2 \\ R_3 & 3 & 4 & 2 & 1 \end{array}.$$

In other words, Dummy takes the numbers on which the permutation acts to be *floating ranks*, whose counterpart is reset at each step.

Let's give another example of Dummy's misinterpretation. The correct iteration of the cycle $(12345) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$ is:

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{array}.$$

The first three rows, in Dummy's version, are on the contrary:

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 4 & 5 & 1 & 2 & 3 \end{array}.$$

Indeed, the first item in the third row is not 3, but 3 as a rank, i.e., as the holder of the third position as determined in the second row, namely 4.

Dummy's heterodox interpretation understands numerals as ranks of positions in the current row.
It also supposes, however, to take numerals to remain self-identical from one row to the next one.
Indeed, remember Dummy's calculation of the third row R_3 of (12345):

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 4 & & & & \end{array}$$

> "2 *becomes* 3*, but since* 3 *now is* 4 *(because* $3.2 = 4$*),* 2 *finally lands on* 4*.*"

In this explanation, 4, which holds the third position in row R_2, remains what it is, namely the numeral 4. In the shift from R_2 to R_3, 4 stays a fixed item, whereas 3 is bestowed the status of floating rank, moving from $3.1 = 3$ to $3.2 = 4$.

This is a double standard that can be objected to.

A more consistent heterodoxical interpretation of permutations consists in identifying as systematically as possible any numeral with a floating rank. Any substitutional matrix $n \rightarrow m$ then reads as a rank updating, which requires rewriting the updated matrix of the substitution at each step. The iteration of the cycle (12345) then goes:

$$
\begin{array}{ccccccccccccccccc}
1 & 2 & 3 & 4 & 5 & & & & & & & & & & & & \\
2 & 3 & 4 & 5 & 1 & = & 1' & 2' & 3' & 4' & 5' & & & & & & \\
4 & 5 & 1 & 2 & 3 & \overset{\rightarrow}{=} & 3' & 4' & 5' & 1' & 2' & = & 1'' & 2'' & 3'' & 4'' & 5'' \\
3 & 4 & 5 & 1 & 2 & \overset{\leftarrow}{=} & 2' & 3' & 4' & 5' & 1' & \overset{\rightarrow}{=} & 5'' & 1'' & 2'' & 3'' & 4'' \\
1 & 2 & 3 & 4 & 5 & = & \leftarrow & \leftarrow & \leftarrow & \leftarrow & \leftarrow & = & 3'' & 4'' & 5'' & 1'' & 2''
\end{array} .
$$

For instance, this is how the first numeral in the fourth row is determined:

> *4 is $3'$. But $3'$ becomes $5'$, i.e., the holder of the 5th position in the row, namely $2'$, which gives, if one gets back to the first five-column group, the numeral 3.*

This calculation is then itself recorded in a third five-column group, as a basis for the calculation of the fifth row.

Obviously, Dummy's heterodox interpretation is wrong (as is its radicalized version).

Indeed, it is based on a confusion of numbers as objects (as when one says that 4 "becomes" 2) and of numbers as contextual ranks in the row of their occurrences (as when one says that 3 has become "the new 2" in the context of the second row of π).

But the point is: Wrong as it may be, Dummy's interpretation is neither trivial nor nonsensical.

This shows that mathematical understanding relies on a correct understanding of how mathematical labellings work.

The permutations of a set are as many symmetries of that set but require, for their handling, the benchmark against which they can appear as symmetries. This benchmark is provided by an arbitrary indexing which acts as a rigidifying device.
This can be generalized.

Labellings, numberings, parameterizations or distinguished elements of various kinds appear to be pervasive throughout mathematics, as local devices to "set ideas."

Let's call settings all devices of that sort.

Settings are actually so pervasive throughout mathematics and so important to grasp that no satisfying account of mathematical objectivity can be sustained without giving them full attention.

There are numerous examples of settings in mathematics.

The writing of a permutation as a two-row matrix is a basic example of what a setting is: It displays a set together with an implicit indexing of it (corresponding to the occurrence ranks of the items in the top row).

Other examples:

- the choice of an origin O for an affine space $\mathcal{E}$, in geometry, leading to the abstract vector space $\mathcal{V}$ underlying $\mathcal{E}$
- the arbitrary choice of a base point to define the fundamental group of a path-connected topological space
- the choice of the particular "atlas" of a differentiable manifold
- the "expansion" of a structure in the model-theoretic sense. (An expansion, however, is less a setting than the internalization of a setting from the semantical metalanguage into the object language.)

Settings correspond to all the mathematically relevant aspects of arbitrary choices in mathematics.

Certain mathematical settings are themselves mathematically formalized as such.

In particular, the choice of an arbitrary origin for the affine space is fully captured by the important mathematical notion of torsor.

Given a group G, a G-*torsor* is a space X on which G acts freely and transitively.

That definition means that X looks exactly like G, except for the fact that there is no distinguished identity element in X.

For instance, the affine space $\mathbb{A}^n$ is a torsor for the corresponding vector space, that is, for the group of translations in $\mathbb{R}^n$.

Other example: the set of orthonormal frames is an $O(n)$-torsor.

A torsor is a group whose identity element is not identified any more: Any element can be chosen to be the identity element.

Conversely, a group G is canonically isomorphic with any of its G-torsors $\mathcal{T}_G$, but only as soon as an element of $\mathcal{T}_G$ has been distinguished to act as the identity:

Until the choice of such distinguished element has been made, $\mathcal{T}_G$ is virtually isomorphic to G, but not canonically so.

For instance, the affine space $\mathbb{A}^n$ equipped with an origin is canonically isomorphic to the group of translations in $\mathbb{R}^n$, but $\mathbb{A}^n$ *as such* is deprived of any group structure.
It is customary to define vectors of $\mathbb{R}^2$ on the basis of the Euclidean plane.
The recovery of a group from one of its torsors is an important example of mathematical setting.

John Baez:

> *Here's a famous example: the set of orthonormal frames at some point of a n-dimensional Riemannian manifold is not the group $O(n)$, but it's an $O(n)$-torsor. You can take any frame and rotate it by an element of $O(n)$; you can take two frames and work out their "difference", which is an element of $O(n)$ – but the frames don't form a group. We can pretend the frames are the group $O(n)$ – but only after we arbitrarily choose one frame and decree it to be the identity. Then every other frame is a rotated version of this one, so we can pretend it is a rotation!*

Back and forth: frames become rotations, rotations are recaptured from frames (as vectors are from points).

Each setting involves arbitrary choices, and so in this sense is "variable" to the extent that it could have been different (without changing the structure that it produces).

But this variability remains only virtual: Once set, a setting cannot change. Otherwise confusions follow (as we have seen).

The confusion of the virtual variability of the parameters of a setting with the actual variability of the objects presented by that setting is the exact root of Dummy's confusion.

The tentation to identify two conjugate permutations
—such as $\begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}$ and $\begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix}$—
illustrates the converse confusion of objects with parameters.

Think also of how the layman gets easily in a muddle about composed percentages or base conversions:

> *If one tries to calculate* 30% *of* 60%*, should one consider that* 30%*, in that context, actually means* 30 *per 60, since* 30% *is meant to apply, not to* 100*, but to* 60%*?*

The confusion has to do with the fact that 100 is just an arbitrary unit to represent ratios and should not be confused with an actual quantity. It is a parameter built into the setting of percentages, instead of an object represented in that setting.

A setting is what endows a structure with a certain angle, so that the structure becomes rigid (i.e., deprived of any nontrivial automorphism). A setting kills all parasitic symmetries.

A setting is the the coordinatization of a mathematical configuration so as to make it mathematically tractable.

Settings are not mere auxiliary devices: They are part of the very objects that they make it possible to manipulate.

Claim: Most mathematical structures are not accessible (= both describable and open to further constructions) without the medium of a particular setting.

Let's call mathematical presentation a mathematical structure as intended through a particular setting.

For instance, the group $\mathfrak{S}_3$ is a mathematical structure for which a relevant presentation is a three-element set $X = \{a, b, c\}$, together with a certain numbering $j : \{1, 2, 3\} \to X$ of its elements.

The case $X = \{1, 2, 3\}$, $j = \mathrm{id}$ is a limit case, still it is a case of mathematical presentation.

Back to the identity problem

A presentation is neither the structure itself, "just as it is," nor a mere particular system instantiating that structure.

A mathematical presentation is not a structure: Its labelling is not canonical and is the tool, not the target of the investigation of the structure.

A mathematical presentation is not a mere system either: If two different settings of the same structure were systems, they precisely could not differ at all.

Three-pronged organization of mathematics: Structures, presentations and systems are irreducible to each other, and equally needed to account for mathematical knowledge *and* mathematical objectivity.

Case study of graph theory

My claim: An unlabelled graph always comes from neutralizing a pre-labelling of the graph (a particular setting of it).

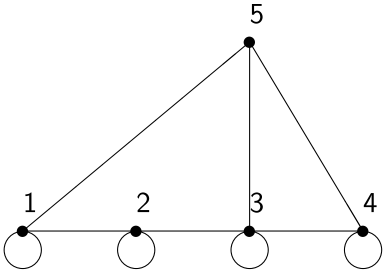
This pre-labelling remains latent despite its neutralization, since it actually underpins its neutralization: The item called ‘b’ could have been called ‘c’, but how to understand this without first giving names?

In a nutshell: Labelled graphs do not come from unlabelled graphs, rather unlabelled graphs come from pre-labelled graphs.

Striking example: Anton Dochtermann, "Hom complexes and homotopy theory in the category of graphs" (2009).

A vertex a of a graph G is said to *dominate* another vertex b of G if all G-neighbors of b are G-neighbors of a.
The operation of erasing a dominant vertex of G is called a *folding* of G.
The converse operation of adding a dominant vertex is called an *unfolding* of G.
A graph G is *reducible* to a graph G' iff G' can be obtained from G by a sequence of foldings and unfoldings.

Now let's consider the following graph G:



Since 3 dominates 4, 4 can be erased. Since 2 dominates 5, 5 can be likewise erased. The vertices 1 and then 2 can be successively erased for the same reason. The graph G appears in the end to be reducible to the single looped vertex 3.
However, G is not reducible to the single looped vertex 4, despite the looped vertices 3 and 4 being rigorously isomorphic, even when considered within G.

This shows how the individuality of certain components of a graph (here, 3) needs sometimes to be taken into account.

Back to the Keränen *vs* Shapiro controversy:

- Shapiro thinks that a presentation still is the corresponding structure.
  At some point of his discussion with Keränen, Shapiro envisages the adjunction of a linear order to a structure **S**, so as to individualize all the places of **S**. He describes this linear ordering as an "enrichment" of **S**.
  But the "enriched structure" is the same structure as **S** (otherwise, it would be *another* structure, and we would not speak of the intended structure any more). So in fact Shapiro makes it appear as though a setting were *internal* to the structure, as though a structure gave by itself the resources to distinguish distinct items in it.

- On the contrary, Keränen thinks that a presentation is already a system: Endowing a structure with a setting suffices to turn it into a mere particular instance of that structure.
  So considering the structure *itself* requires to get rid of any means to distinguish numerically distinct items.

Shapiro confuses presentations with structures, while Keränen confuses them with systems.
As a result, the identity problem has led to an unending debate.

The mistake shared by both Shapiro and Keränen is to fail to distinguish between structures, systems *and presentations*.

(That being said, the distinction between structures and presentations is a relative one: In some cases, a structure may appear to be a presentation w.r.t some more abstract structure.)

Solution to the identity problem

As soon as the intermediate level of settings and presentations is taken seriously, the identity problem dissolves: Any nontrivial automorphism of a given structure actually is an isomorphism between two (rigid) presentations of that structure, and can be introduced only as such.

Symmetries do exist, but their introduction presupposes the very distinction of the items that they make indiscernible.
This distinction is a distinction within a presentation: for instance, between i and $-i$ within $\langle \mathbb{C}, i, -i \rangle$, and, consecutively, a distinction between two settings (between $\langle \mathbb{C}, i, -i \rangle$ and $\langle \mathbb{C}, -i, i \rangle$ (the latter referring to the former).

Since the chosen presentation presents the structure $\mathbb{C}$ itself, this solution to the identity problem is compatible with structuralism.

But this solution comes at a cost: the cost of acknowledging that, in many cases, a mathematical structure is not accessible "as it is."

Natural question: Since presentations are not structures, *what is an isomorphism between presentations?*

Answer: Isomorphisms between presentations are presented isomorphisms, i.e., isomorphisms in the usual sense.

Indeed, isomorphisms always implicitly refer to the choice of particular settings: How, for instance, to define an isomorphism between $\mathbb{R}[X]/(X^2 + 1)$ and $\mathbb{R} + i\mathbb{R}$ without mentioning distinguished elements X and i?

Admittedly, in certain cases the mere existence of an isomorphism suffices ("Two isomorphic sets have the same cardinality").
But in the other cases, the isomorphism needs to be specified (important example: Whitehead's Theorem).

If one does not distinguish between structures and presentations, then any isomorphism between two structures shows, by its very existence, that the two structures actually are identical, so that they cannot really be said to be merely isomorphic.
This is but a variant of the identity problem.

On the contrary, if one does distinguish between structures and presentations, then any isomorphism between two presentations simply shows that the latter actually are two different presentations of a single structure.

Most automorphisms are isomorphisms between two presentations, whose respective underlying structures can *then* be considered —only derivatively to that isomorphism— as corresponding to a single structure.

Thus, the notion of presentation-isomorphism is in fact more primitive and more robust than the notion of structure-isomorphism.

To sum up

The identity problem confronts the realist structuralist with the fact that an object seems to be more than a place in a structure, namely a place plus something that individualizes it within its indiscernibility class.
This "something more" is nothing but a setting.

A setting is not part of the corresponding structure itself, yet it is not a merely psychological apparatus: It is really part of the structure, to the extent that it can be manipulated and analyzed.

Settings and presentations do not boil down to sub-mathematical conditions of concrete mathematical activity, such as the actual drawing compared to the geometrical theorem. When they occur, they are integral to mathematical understanding *and* to mathematical objectivity.

Permutations, graphs, mathematical objects in general have to be considered through the actual ways in which only *it becomes possible* to grasp and manipulate them.

The epistemic handleability conditions of mathematical objects should be built into mathematical objectivity.

Settings are precisely those parameterizing devices without whose mediation an intended structure, in many cases, at least, cannot be reached, i.e., introduced and cognitively handled.

Understanding (at least some) mathematical misunderstandings

Our main example has been Dummy's misunderstanding.

This misunderstanding follows a pattern: In most cases, the misunderstander is not aware of the distinction to be made between the parameters and the targets of a particular setting, since no setting has been explicitly introduced in the first place.

And the mathematician is unable to understand the misunderstander's confusion, since the grasp of the setting is a pre-condition to having any mathematical question to ask to begin with.

The misunderstanding of mathematical misunderstandings is a serious shortcoming, not only for any philosophy purporting to account for mathematical practice, but also for any philosophy purporting to account for mathematical objectivity.

Shapiro's ontological platonism implies to dismiss the epistemic conditions that make it possible to handle mathematical objects.

But epistemology strikes back, in the form of the identity problem.

The identity problem actually is the symptom of something more general, which explains its being analogous to Benacerraf's dilemma.
One indication of the analogy is the common stress put on singular terms by Benacerraf's platonist and by Shapiro.

Getting back to Benacerraf's "Mathematical truth:" The epistemic horn is the wrong thesis that mathematical presentations do not present anything;
The semantic horn —or *ante rem* structuralism— is the wrong thesis that presentations are mere artefacts as opposed to the structures "in themselves."

The common root of both puzzles (Benacerraf's dilemma, the identity problem) is the uncritical acknowledgement of the surface grammar of mathematical language and, accordingly, the myth of the simple intuitability of mathematical items "just as they are," were they structures (to the platonist) or signs (to the formalist).

The spectrum of mathematical presentations is hardly amenable to a single kind, but a mathematical presentation always relies ultimately on the use of notations in an environment that shows how shifting from a notational system to another would be possible but incidental to what is at stake.

The knowledge of a mathematical presentation is not a simple intuition, but a double understanding: that both of a setting and of the possible reparameterization of it (without anything relevant being changed).

A presented structure is the abstract form of all its possible instances, whereas it is the invariant of all its possible presentations.

Anything established about the presentation of a structure is true of that structure; The same, of course, does not hold for a system.

This is in particular why, once again, the connection between a structure and any of its presentations is something entirely different from the connection between a structure and any of its instances.

Unearthing a presentational tradition

The concept of presentation has both a mathematical side and a philosophical side.

WARNING. One can trace back each side to particular precedents, but those precedents broaden the concept of presentation toward the (unsufficiently specific) notion of "way of defining" a mathematical object.

So there is a tradition to unearth, but this tradition will then have to be delineated as sharply as possible, and refined so as to illustrate the concept of presentation instead of some other too broad notion. This task goes beyond the scope of this talk.

Mathematical side:

- all systems of parameterization already discussed
- the presentation of a group: $[a, a^n = 1] = \{a^k : k \in \mathbb{N}\}/(a^n)$ is (a presentation of) the cyclic group $\mathbb{Z}/n\mathbb{Z}$.
- the projective resolution of a module.

Philosophical side:

- Brentano: "Every intentional experience is either a presentation (*Vorstellung*) or is founded upon a presentation."
- Husserl, *Fifth Logical Investigation*: Husserl objects to the idea that, underlying every consciousness, there are acts of a special kind which merely present objects as the content for other, higher-order acts of consciousness.
- Frege: The very phrase "mode of presentation" [*Art des Gegebenseins*] is mentioned by Frege in the context of the distinction that he drew between sense and denotation, in order to account for the nontrivial nature of mathematical identities.

Conclusion (1/2)

Main points and advantages of a presentational perspective:

1. Mathematical settings and presentations are essential components of both mathematical knowledge and mathematical objectivity.
2. The "identity problem" stems from the confusion of mathematical presentations with either structures or systems, and on the contrary is solved by the corresponding distinctions.
3. The solution also applies to Benacerraf's dilemma.
4. Mathematical understanding relies in particular, in many cases, on a correct understanding of the behavior of mathematical settings. This understanding is a double understanding, not a simple intuition (whatever its object may be).

Conclusion (2/2)

5. Mathematical understanding should be understood so as to make it possible to understand mathematical misunderstandings. (Dummy is not a freak.) This is what a presentational account of mathematical knowledge endeavors to do.
6. A presentational account aims to ensure the unity of mathematical knowledge and mathematical objectivity as being something more than a fortunate match.
7. A presentational perspective simply gathers many suggestions. But it maybe has the benefit of trying to bring together philosophy of mathematical practice, "classical" philosophy of mathematics, and maybe general philosophy.