

Logical universality and logical contextuality in Frege

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Basic working hypothesis

Some philosophical ideas and logical devices in Frege and Russell are still to be reactivated.

Logic has made historical choices, but this does not mean that no other choices could be made. Frege and Russell did not do certain things, but that does not mean that they could not have done them.

The reactivation of some aspects of Frege's and Russell's logic ties in with the reconsideration of the issue of "absolute generality" (Richard Cartwright, Timothy Williamson, Agustìn Rayo, Gabriel Uzquiano, Gregory Landini, ...).

In this talk I will argue that Frege's universalism should not be conflated with a stiff absolutism: the defense of the universality of logic is compatible with the admission of variable contexts of discourse.

Main lines of my talk

- ▶ First, compare the respective ways in which Kant and Frege combine two different features commonly ascribed to logic.
- ▶ Secondly, trace back the label “logical universalism” to its root, and criticize it.
- ▶ Thirdly, show how Frege’s semantical framework can countenance particular contexts of discourse and sketch a formal framework to illustrate that point and compare Frege with Tarski.
- ▶ As a consequence, show (*contra* Hodges and Demopoulos) that Frege *can*, as well as Tarski, make sense of non-logical constants as “indexicals”.

Frege's "logical universalism"

Logic is a theory that precedes any other theory. The jurisdiction of logic knows no bound. This traditional characterization of logic is cashed out in terms of two distinct features:

- ▶ the **universality** of logic
- ▶ the **radicality** of logic

The universality of logic consists in its being about absolutely everything : only one type of entity variables ; a single unrestricted range of values for entity variables. **Nothing can escape logic.**

The radicality of logic corresponds to there being the one and the same logic that any reasoning must comply with. The principles of logic are laws that one cannot but presuppose them, even to challenge them. **Nobody can escape logic.**

The universality and the radicality of logic have been associated since at least Kant's *Logic*:

If [. . .] we put aside all cognition that we have to borrow from objects and merely reflect on the use just of the understanding, we discover those of its rules which are necessary without qualification, for every purpose and without regard to any particular objects of thought, because without them we would not think at all. ("The Jäsche logic", Ak. ix, 12)

Logic is the "science of necessary laws of thought, without which no use of the understanding or of reason takes place at all" (Ak. ix, 13). **(Universality)**

It is, as well, "a science a priori of the necessary laws of thought, not in regard to particular objects, however, but to all objects in general" (Ak. ix, 16). **(Radicality)**

Frege himself gives in sometimes to a kind of confusion:

Here, we have only to try denying any one of them [one of the fundamental truths of arithmetic], and complete confusion ensues. Even to think at all seems no longer possible. The basis of arithmetic lies deeper, it seems than that of any of the empirical sciences, and even than that of geometry. The truths of arithmetic govern all that is numerable. This is the widest domain of all; for to it belongs not only the actual, not only the intuitable, but everything thinkable. Should not the laws of number, then, be connected very intimately with the laws of thought? (Foundations of Arithmetic, § 14)

John MacFarlane: Kant and Frege do not conceive of the generality of logic in the same way. Kant sees logic as a canon (a body of rules), whereas Frege sees it as a science (a system of truths). To Kant, logic is contentless, whereas to Frege it has a content with the widest scope.

As a consequence, Kant insists on the radicality of logic, whereas Frege insists on its universality.

MacFarlane: Despite the discrepancies, Kant and Frege concur in characterizing logic through its normative generality.

Logical laws, as Frege describes them, “are the most general laws, which prescribe universally the way in which one ought to think if one is to think at all” (*Grundgesetze*, xv). The special status of logic comes from the generality in the applicability of the norms it provides.

MacFarlane's analysis somewhat conceals the asymmetry of Kant's and Frege's respective agendas.

To Kant, radicality is a way of accounting for formal universality from within a nonformal conception of logic.

The domain ruled by formal logic can absolutely not be seen as more inclusive than the domain ruled by transcendental logic.

The generality of formal logic has to be reckoned through something else than its universality, namely: it states conditions of possibility of thought as such.

Frege's move goes the other way around: the universality of logic allows Frege to express the radicality of logic from within an anti-psychologistic conception of logic, without committing himself to some slippery notion of necessity.

To sum up: Both Kant and Frege bestow on logic two distinctive features: universality and radicality.

And both agree to consider those features as inseparable.

But Kant puts the stress on radicality,

whereas logical universality, in Frege, prevails over logical radicality.

Logical universalism is more than the association of logical universality and logical radicality.

It is the sheer conflation of both, and the further claim that “one cannot consider one’s language from the outside”.

Although a label that has been pinned on to Frege, it is *not* epitomized by Frege himself.

Logical universalism is more of a reconstruction by authors who fought a family of philosophical stances of authors such as Frege, Russell, Wittgenstein and, to some extent, Carnap and even the young Tarski.

A tradition initiated by van Heijenoort put in the same “universalist” bag four independent claims:

- ▶ logic resorts to absolutely unrestricted (unspecified) variables
- ▶ logic refers to one single universe of discourse
- ▶ the language of logic is unescapable
- ▶ metatheoretical studies, and semantics to begin with, are impracticable and, for that reason, banned.

Another important consequence of the universality of logic is that nothing can be, or has to be, said outside of the system. And, in fact, Frege never raises any metasystematic question (consistency, independence of axioms, completeness). [...] Questions about the system are as absent from Principia Mathematica as they are from Frege's work. Semantic notions are unknown. [...] The universal formal language supplants the natural language, and to preserve, outside of the system, a notion of validity based on intuitive set theory, does not seem to fit into the scientific reconstruction of the language. ([van Heijenoort(1967)], p. 326-327)

Hintikka about the “ineffability of semantics”:

One of the main consequences of the universality of language (universality of the language) is that I cannot in my language speak of how its semantical relations to the world could be changed, at least not in a large scale. But such a systematic variation of the interpretation of a language is what the model theory for this language is all about. To speak of different models of a theory or a language in a logician's sense is ipso facto to speak of different systems of referential relations (interpretations) connecting language (or a part thereof) with the world. Hence all model theory is impossible according to the strict constructionist version of the universalist assumption. [Hintikka(1997)], p. 216)

There is but a single interpretation of language because there is but a single universe of discourse because there is but a single world, and conversely:

Speaking of other possible worlds presupposes a modicum of change in the references in our terms and other expressions, and hence some amount of faith in the conception of language as a reinterpretable calculus. For this reason, the so-called possible worlds semantics, and its predecessor, model theory for modal logics, cannot be acceptable to a true universalist.

[. . .]

A believer in the free reinterpretability of our language would answer a question concerning the range of an existential quantifier by saying, 'Whatever we have included in the relevant universe of discourse.' For a universalist, there is only one range for one's (first-order) quantifiers, viz., all the individual objects in the world. (Op. cit., p. 218-219)

The universality of logic does not preclude one from considering local universes of discourse, in particular for metatheoretical purposes.

- ▶ *Principles* already, §430: Russell was well aware that an axiomatic system (Dedekind's arithmetic) may lend itself to various different interpretations.
- ▶ Russell's perspectivist reconstruction of the external world
- ▶ Philippe de Rouilhan ([de Rouilhan(2012)], p. 573-574): Russell, and Frege as well, did not, but *could have* introduced models of a theory within a universal logical system (a system based on the use of universal variables) and thus *could have* built up, within such a system, the equivalent of Tarskian model theory.

Frege, “On the Foundations of Geometry” (1903):

[. . .] Euclidean geometry presents itself as a special case of a more inclusive system which allows for innumerable other special cases – innumerable geometries, if that word is still admissible. [. . .] If one wanted to use the word “point” in each of these geometries, it would become equivocal. To avoid this, we should have to add the name of the geometry, e.g. “point of the A-geometry,” “point of the B-geometry,” etc. Something similar will hold for the words “straight line” and “plane.” ([Frege(1971)], p. 37)

Obviously, all special geometries are not subdomains of a single geometrical domain, but correspond to different theoretical contexts.

So Frege was not insensitive to the idea of context variation.

Origin of the confusion of universality and radicality: the post-scriptum (1935) of Tarski's essay on "The concept of truth in formalized languages".

Tarski's main result: An adequate definition of the truth predicate of some language can only be given in a metalanguage whose order is strictly greater than the order of the language under consideration. Notion of order: linked to that of "semantical category".

Post-scriptum: Tarski shifts to a new notion of order, drawn from the cumulative hierarchy of ZF set theory.

This change precipitated a set-theoretical conception of the hierarchy of languages: the domain of the metalanguage must be strictly wider than the domain of the object-language.

As a consequence, a universal domain cannot lend itself to any truth theory. But this follows only from an artefact of Tarski's analysis.

Logical universality does not block the development of semantical considerations.

Frege's universalism can be loosened up, so as to make sense of contextuality.

Precisely, Frege's work on the foundations of geometry, in the context of independence proofs in geometry, shows that the existence of an absolute, universal domain of quantification does not prevent one from introducing various systems of thoughts.

Variable reinterpretations as internal translations:

Imagine a vocabulary: not, however, one in which words of one language are opposed to corresponding ones of another, but where on both sides there stand words of the same language but having different senses. [...] We may say in general that words with the same grammatical function are to stand opposite one another. Each word occurring on the left has its determinate sense—at least we assume this—and likewise for each one occurring on the right. Now by means of this opposition the senses of the words on the left are also correlated with the senses of the words on the right. Let this correlation be one-to-one, so that on neither the left nor the right is the same thing expressed twice. We can now translate; not, however, from one language to another, whereby the same sense is retained; but into the very same language, whereby the sense is changed. ([Frege(1971)], p. 107-108)

Definition of dependence:

Let us now consider whether a thought G is dependent upon a group of thoughts Ω . We can give a negative answer to this question if, according to our vocabulary, to the thoughts of group Ω there corresponds a group of true thoughts Ω' , while to the thought G there corresponds a false thought G' . For if G were dependent upon Ω , then, since the thoughts of Ω' are true, G' would also have to be dependent upon Ω' and consequently G' would be true.

With this we have an indication of the way in which it may be possible to prove independence of a real axiom from other [real] axioms. ([Frege(1971)], p. 110)

Important difference with Tarski: A Fregean vocabulary is a correspondence $e \rightarrow e'$, where e and e' are expressions from the same interpreted, logically regimented language L . It is not an arbitrary map from one domain (L -structure) to another.

Logical contextuality

Frege's perspective is neither in principle, *nor in fact* incompatible with the recognition of various contexts of discourse.

Further claim: Wilfrid Hodges and William Demopoulos are wrong to diagnose Frege as being unable to account for non-logical constants of a syntactically rigorous language, as well as to account for indexicals in natural language.

Frege affords the means to account for non-logical constants, and thus nothing precludes him from accounting for indexicals along the same lines.

Hodges suggested that non-logical constants of a formal language can be seen as indexicals within the space of structures for the language.

For example, the symbol \cdot of the group operation, in the language L of group theory, is interpreted, in the *context* of each L -structure, by a specific operation in that structure: in any group G , “[...] the symbol \cdot automatically refers to the operation labelled \cdot in the group G ” (p. 150).

Non-logical constants are given a reference by being applied to a particular structure.

The conceptual payoff is that “there is nothing *sui generis* about truth in a structure” (*ibid.*). Truth in a structure is just truth, being granted the semantical resources of contexts for the interpretation of non-logical constants as indexicals.

To Frege and Peano, a rigorous language is a logically regimented fragment of natural language.

A sentence of a “Frege-Peano language” is interpreted from the outset, and thus is in itself true or false.

According to Demopoulos, Frege’s blindness to non-logical constants would stem from his blindness to indexicals, which itself would stem from his blindness to *contextuality*:

One of the features of natural language that simply has no analogue in a Frege-Peano language is the presence of indexicals. Non-logical constants, being like indexical expressions in the way they function, would for this reason alone make it difficult to express Huntington’s axiomatization within a Frege-Peano language [...] for Frege the sense of a designating expression determines its reference ‘absolutely’, i.e. independently of such contextual considerations as are relevant in the case of non-logical constants. ([Demopoulos(1994)], p. 216-218)

I now would like to challenge this kind of assessment.

Let L be a fixed language, a sublanguage of “the” language L_0 (for instance, L is the Frege-Peano version of the language of group theory),

and let σ be the signature of L (in the case of group theory, $\sigma = \{e, \cdot, (-)^{-1}\}$).

A *Fregean σ -structure* S consists of:

- ▶ a set Ω of expressions, together with a “vocabulary” $f : \Omega \rightarrow \Omega_0$ such that $\sigma \subseteq \text{Im}(f)$, where Ω_0 is the set of all expressions of the language L_0
- ▶ a set T of thoughts expressed with the resources provided by Ω .

Each σ -structure can thus be understood of as a context for the use of all the elements of σ , themselves seen as indexicals. The theory T are all the thoughts asserted to be true in the σ -structure. In case $f(c) = c$, S is said to adopt the intended interpretation of ‘ c ’.

A morphism $S = \langle \Omega, T \rangle \rightarrow S' = \langle \Omega', T' \rangle$ of Fregean σ -structures is a vocabulary $g : \Omega \rightarrow \Omega'$ such that:

- ▶ the diagram $\Omega \xrightarrow{f} \Omega_0$ commutes;

$$\begin{array}{ccc} \Omega & \xrightarrow{f} & \Omega_0 \\ g \downarrow & \nearrow f' & \\ \Omega' & & \end{array}$$

- ▶ for each $t \in T$, $g(t)$ or $\neg g(t) \in T'$.

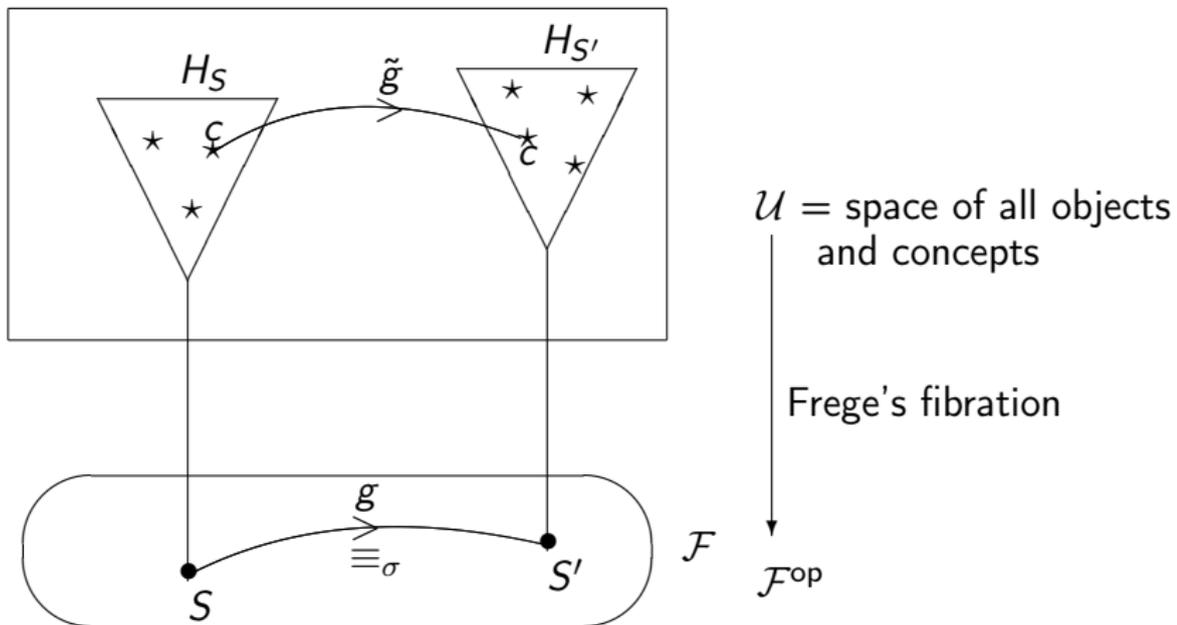
Let \mathcal{F} be the category of Fregean σ -structures thus defined.

A σ -structure S is σ -congruent with a σ -structure S' , written $S \equiv_{\sigma} S'$, iff there is a morphism $S \rightarrow S'$ of σ -structures such that $\sigma \subseteq \text{Fix}(g)$.

The relation of σ -congruence (or, at least, its symmetrical closure) is an equivalence relation.

A *context* is a σ -congruence class of σ -structures, for some implicit signature σ .

It is a complete class of σ -structures sharing the same re-interpretation of all members of σ .



Frege's model theory

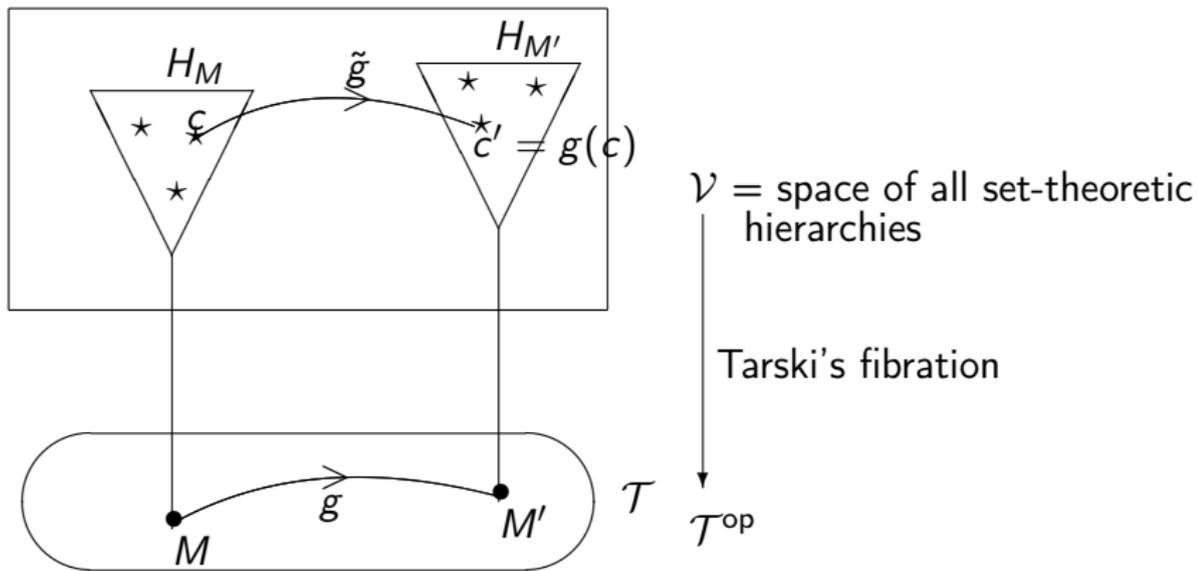
In the picture above, H_S is the hierarchy of objects and concepts (of all levels) involved in T , namely the collection all the objects and concepts to which all thoughts in T refer.

\tilde{g} is the map from H_S to $H_{S'}$ induced by g .

The two structures S and S' are supposed to be σ -congruent, so each non-logical constant c in the signature σ is preserved by the corresponding σ -congruence g .

The context to which both S and S' belong is their common σ -congruence class: it consists in the web of all the σ -congruences such as g .

All σ -structures of the same σ -congruence class as S can be seen regarded as alternative possible worlds in that context.



Tarski's model theory

In the picture above, H_M is the set-theoretic hierarchy of definable subsets based on $|M|$, and \tilde{g} is the map from H_M to $H_{M'}$ induced by g . The map g is supposed to be a σ -homomorphism.

Let \mathcal{T} be the category of all σ -structures and σ -homomorphisms.

Remark: In Tarski's semantics, *each* σ -structure M is a context of its own. But M is the unique possible world that can be considered in its own context (in context M).

Otherwise put, only diagonal or "primary" intentions can be represented.

From Tarski's perspective, it does not make much sense anyway to consider different σ -structures in the same particular context.

One last remark about the two pictures

In Frege's model theory, the correspondence $S \mapsto H_S$ is functorial, which allows one to define a "fibration" $p_F : \mathcal{U} \rightarrow \mathcal{F}^{\text{op}}$ whose base category is \mathcal{F}^{op} .

In the same way, in Tarski's case, the correspondence $M \mapsto H_M$ allows one to define a fibration $p_T : \mathcal{V} \rightarrow \mathcal{T}^{\text{op}}$ over \mathcal{T}^{op} .

In both cases, a non-logical constant becomes a *section* of the fibration under consideration. A section is the choice, above *each* point of the base, of a particular entity in the corresponding fiber, so that all the choices thus made comply with the conditions expressed by the morphisms of the base.

Of course, this is almost tautological, since the signature σ is precisely built into those morphisms.

Still, it allows to picture and highlight non-logical constants as being indexicals.

The *role* (or the *character*) of a demonstrative, says Perry, is a “rule that takes us from each context of utterance to a certain object”. In the case of a non-logical constant (seen as an indexical), the rule at stake corresponds to the constraint set upon the possible choices for a section of the fibration.

A non-logical constant, in the Fregean setting, as well as in the Tarskian one, consists of successive choices (along the morphisms of the base category) that have to make up a *section*.

Two conclusions:

- ▶ Understanding non-logical constants as indexicals is not beyond Frege's logical horizon.
- ▶ A non-logical constant, understood as a section, does not work in the same way at all in Frege and in Tarski. *A change of context, in Frege, is not a change of place, but an internal reorganization of language.*

Language is not a mere collection of words, and thus any change is always a change of several interconnected meanings. By comparison, each non-logical constant, in Tarskian semantics, can be reinterpreted independently of the rest of the language.

Three claims by way of conclusion

In his first *Logical Investigation*, Frege seems to be bogged down by the contextuality elicited by demonstratives such as “I”.

Demopoulos’ diagnosis, following Hodges, is that contextuality is, as such, downright foreign to Frege’s conception and practice of logic.

Perry basically concurs with Demopoulos.

First claim. The emphasis laid on demonstratives and the sensitivity to contextuality evinced by Frege’s first *Logical Investigation* are mainly to be traced back to Frege’s reaction to Hilbert’s axiomatic method in mathematics.

The treatment of non-logical constants can be the right starting point to reconsider Frege’s predicament about demonstratives in the first *Logical Investigation*.

However, the framework presented above does not come close to sketching how proper indexicals could be really accounted for along Fregean lines.

Second claim. Logical universality is perfectly compatible with contextuality.

Believing that logical universality excludes contextuality amounts to confusing a context variation with a domain variation. But a context variation is not a domain variation.

It is not true that Frege *could not* make any sense of the very idea of a context.

It is the other way around: Frege's model theory makes it possible to define contexts, whereas Tarski's semantics conflates contexts with domains and merges both notions into the mixed notion of "universe of discourse".

Third claim.

Variable contexts could be built up from within the Fregean framework of an interpreted language.

Non-logical constants and indexicals are both kinds of sections. And Fregean sections do not work in the same way as Tarskian sections.

At least, variable contexts can be emulated from a Fregean perspective, and this is sufficient a reason to loosen up the logical universalism often attached to Frege.



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