Category theory and set theory: algebraic set theory as an example of their interaction

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My talk will be devoted to an example of positive interaction between (ZFC-style) set theory and category theory, namely Algebraic Set Theory (AST).

I will focus on the first three axioms of the original axiomatization of AST by André Joyal and Ieke Moerdijk.

I will argue that AST can be described **as a genuine graft of the theory of fibered categories onto ZFC**, which goes beyond the unfruitful opposition often established between set theory and category theory.

Championing either Set Theory or Category Theory is just extrapolating a tradition within mathematics (analysis and logical semantics in the case of Set Theory, algebraic geometry and modern algebraic topology in the case of Category Theory).

“Foundations” are provided neither by ST nor by CT.

It is better to try to go beyond the contest.

Presentation of AST

Two different approaches : a logical one (Alex Simpson, Steve Awodey), a geometrical one (Joyal-Moerdijk).

Joyal & Moerdijk: generalization of models of ZF ("free ZF-algebras").

Awodey & co: general program of completeness results between different axiomatizations of set theory and different collections of "categories of classes."

Let C be a Heyting pretopos, which means that C is rich enough to interpret first-order logic and arithmetic. (Still, C is not supposed to be a topos, and in particular to have "power objects.")

The basic idea is to characterize a special subcollection of the collection of all maps of C: a class S of "small maps" in C.

Intuitively, **an arrow is a small map if all its fibers have a set-like size.**

Then, an object X of C is said to be *small* if $X \to 1$ is a small arrow.

Upshot: Arrows are brought to the fore, instead of sets, and sets themselves are conceived of as "fibers."

Axioms for the class S:

1. Any isomorphism of C belongs to S and S is closed under composition.
   This axiom corresponds to the fact that the union of a small family of small sets is a small set.
2. Stability under change base: for any pullback

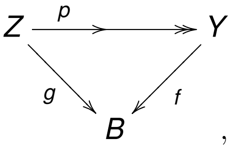
$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow[p]{} & X \end{array},$$

   if f belongs to S, so does g.
3. "Descent": for any pullback along an epimorphic arrow p

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \overset{}{\underset{p}{\twoheadrightarrow}} & X \end{array},$$

   if g belongs to S, so does f.

4. The arrows $0 \to 1$ and $1 + 1 \to 1$ belong to S.
   This axiom ensures that the empty set (initial object) is small and that any finite set is small.
5. If two arrows $f : Y \to X$ and $f' : Y' \to X'$ belong to S, then so does their sum $f + f' : Y + Y' \to X + X'$.
   This axiom ensures that the disjoint union of two small sets (objets) is a small set.
6. In any commutative diagram



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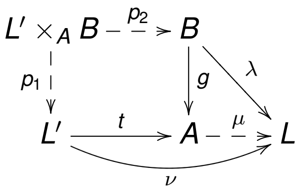
if g belong to S, then so does f.

If L is a partially ordered set ("poset") in C, then $\mathrm{Hom}_C(A, L)$ is itself a poset.

For $g : B \to A$ and $\lambda : B \to L$, the *supremum of* λ *along* g is a map $\mu : A \to L$ s.t., for any $t : L' \to A$ and any $\nu : L' \to L$,



$$\lambda \circ p_2 \leq \nu \circ p_1 \text{ in } \mathrm{Hom}_C(L' \times_A B, L) \;\text{ iff }\; \mu \circ t \leq \nu \text{ in } \mathrm{Hom}_C(L', L)$$

L is said to be *S-complete* if any map to L has a supremum along any map in S.

Definition: A $\mathtt{ZF}$-*algebra* in $\mathcal{C}$ is an $\mathcal{S}$-complete poset L with all joins, endowed with a map $s : L \to L$.

The partial order $\leq$ of L corresponds to inclusion, and the map s corresponds to the singleton operation.

Given a $\mathtt{ZF}$-algebra (L, s), it is possible to define a "membership relation" $\epsilon \rightarrowtail L \times L$ on L, by setting: $x \, \epsilon \, y$ iff $s(x) \leq y$ for any x and y "in" L.

A notion of homomorphism of ZF-algebras in C is readily available.

Fact: Any object A of C generates a *free* ZF-*algebra* in C, written $V_{\langle C,S\rangle}(A)$.

$V_{\langle C,S\rangle}(A)$ is the (generalized) cumulative hierarchy on A.

Fact: Adding two further mild assumptions about S, one gets:

$$A + V_{\langle C,S\rangle}(A) \text{ is a model of ZF set theory with atoms.}$$

The main purpose of my talk is to explain the elliptic phrase "Descent" used to describe the third axiom of AST.

Descent theory is a central tool of abstract algebraic geometry, coming from Grothendieck's work.

Descent theory makes sense only in the general context of "fibered categories," a.k.a. "fibrations."

So I will explain:

- what fibrations are
- what descent consists in
- the rationale of AST's first three axioms.

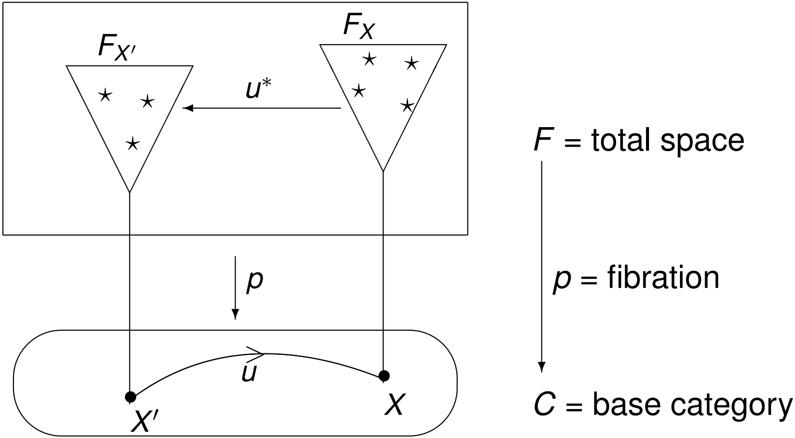
Fibrations

Fibration (fibered category) = *category-theoretic generalization of the notion of surjective map*.

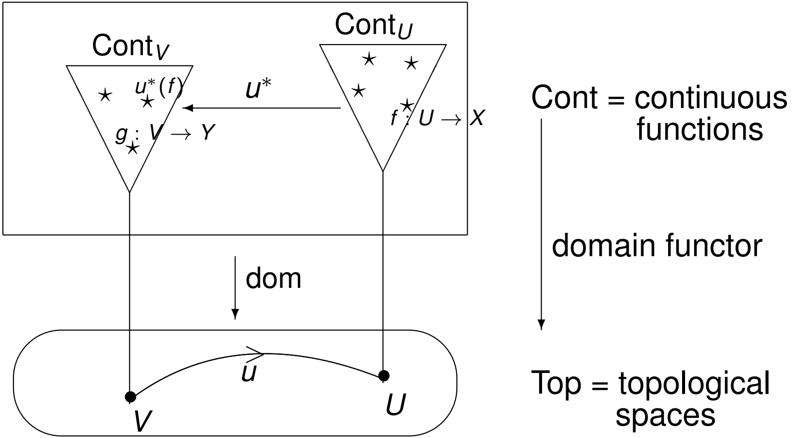
A *fibration* is a functor $p : F \to C$.

For each C, $p^{-1}(X)$ is a category written F_X , and called the "fiber above X."

Each arrow $u : X' \to X$ in C gives rise to a "base change" functor $u^* : F_X \to F_{X'}$ between the corresponding fibers (in the reverse direction).



A fibration in general



For $f : U \to X, \;\; \mathbf{u}^*(\mathbf{f}) = \mathbf{f} \circ \mathbf{u} : V \to X$

EXAMPLE of fibration

Given a category C, let $C^{\rightarrow}$ be the category of all arrows in C. **The codomain functor cod** $: C^{\rightarrow} \rightarrow C$ **is a fibration.**

The fiber above each $X \in \text{Ob } C$ contains all the arrows in C with codomain X.

The class S **of small maps is thought of as a sub-fibration** $S \rightarrow C$ **of the codomain fibration cod** $: C^{\rightarrow} \rightarrow C$**.**

Descent theory

Descent theory = abstract framework geared to describe glueing processes, i.e. the shift from local data to a global item.

TYPICAL EXAMPLE.
Let $(U_i)_{i \in I}$ be a covering of a space U, and suppose that for each $i \in I$ a continuous function $f_i : U_i \to V$ is given, in such a way that

$$\forall i, j \in I \;\; f_i|U_{ij} = f_j|U_{ij},$$

where $U_{ij} := U_i \cap U_j$.

Then there exists a unique function $f : U \to V$ s.t. $\forall i \in I \;\; f|U_i = f_i$.

Reformulation in the context of dom : Cont → Top

Let's write $U' := \coprod_{i \in I} U_i$. The canonical map $\alpha : U' \to U$ is both continuous and surjective.
And the family $(f_i)_{i \in I}$ exactly constitutes an object in $\mathrm{Cont}_{U'}$.

Let's introduce the following pullback:

$$\begin{array}{ccc} U' \times_U U' & \overset{p_2}{\dashrightarrow} & U' \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \alpha} \\ U' & \xrightarrow[\alpha]{} & U \end{array}$$

$U' \times_U U' \simeq \coprod_{i,j \in I} U_{ij}$

For $x \in U_{ij}$, $p_1(x) = x \in U_i$ and $p_2(x) = x \in U_j$

For $(f_i)_{i \in I} \in \mathrm{Cont}_{U'}$:

- $p_1^*((f_i)_i) = (f_i|U_{ij})_{i,j}$
- and $p_2^*((f_i)_i) = (f_i|U_{ji})_{i,j} = (f_j|U_{ij})_{i,j}$.

So the hypothesis $\forall i,j \in I \;\; \phi_{ij} : f_i|U_{ij} = f_j|U_{ij}$ becomes:

$$p_1^*((f_i)_i) = p_2^*((f_i)_i)$$

$$U' \times_U U' \qquad\qquad U' \qquad\qquad U$$

$$p_1^*((f_i)_i) = p_2^*((f_i)_i) \text{ in } \mathrm{Cont}_{U' \times_U U'}$$

$$U' \times_U U' \qquad U' \qquad U$$

$$(U_i \cap U_j)_{i,j} \qquad (U_i)_i \qquad U$$

More generally, replacing identity with an isomorphism, one needs an isomorphism

$$\phi : p_1^*((f_i)_i) \simeq p_2^*((f_i)_i) \text{ in } \mathrm{Cont}_{U' \times U'},$$

i.e. a family

$$\phi_{ij} : p_1^*(f_i) \simeq p_2^*(f_i)$$

of isomorphisms in the fibers above the U_{ij}'s (those isomorphisms being compatible when they overlap).

You can glue the local data above the components U_i of the covering U', so as to get a global item above $U = \bigcup_i U_i$.

Grothendieck, “Technique de descente et théorèmes d’existence en géométrie algébrique” (1959).

Let F be a fibration with base category C.
Recall that any arrow $f : X'' \to X'$ in C gives rise to a functor $f^* : F_{X'} \to F_{X''}$.

Given an arrow $\alpha : X' \to X$ in C and $\xi' \in F_{X'}$, a **descent datum on $\xi' \in F_{X'}$ w.r.t. α** is an isomorphism

$$p_1^*(\xi') \simeq p_2^*(\xi') \text{ in } F_{X' \times_X X'}$$

satisfying compatibility conditions:

$$p_1^*(\xi') \simeq p_2^*(\xi') \qquad \xi' \qquad ?$$

$$X' \times_X X' \underset{p_2}{\overset{p_1}{\rightrightarrows}} X' \xrightarrow{\quad \alpha \quad} X$$

An arrow $\alpha : X' \to X$ in C is a *strict F-descent morphism* if:

(A) $\alpha p_1 = \alpha p_2$, and for any two objects ξ, η of F_X, $\alpha^* : F_X \to F_{X'}$ induces a 1-1 map from $\mathrm{Hom}_{F_X}(\xi, \eta)$ to the subset of $\mathrm{Hom}_{F_{X'}}(\alpha^*(\xi), \alpha^*(\eta))$ consisting of all the arrows $v : \alpha^*(\xi) \to \alpha^*(\eta)$ such that $p_1^*(v) = p_2^*(v)$

(B) giving an object of F_X exactly amounts to giving an object of $F_{X'}$ equipped with a descent datum w.r.t. α.

"What should glue actually glues along α" (see *infra*).

$p_1^*(v) = p_2^*(v)$ means that all the components of v have identical restrictions to $X' \times_X X'$.

$$\begin{array}{ccc} & \alpha^*(\xi) & \xi \\ p_1^*(v) \downdownarrows p_2^*(v) & \Big\downarrow v & \Big\downarrow u \\ = & \alpha^*(\eta) & \eta \end{array}$$

$$\begin{array}{ccccc} X' \times_X X' & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & X' & \xrightarrow{\quad\alpha\quad} & X \\ & & & & \\ (U_i \cap U_j) & & (U_i) & & (U) \end{array}$$

Condition (A) says that in that case the components of v can actually be glued together along α, so as to come from some arrow u in F_X:

$$v = \alpha^*(u)$$

A descent datum is a family $\xi' \in F_{X'}$ of objects whose respective restrictions in $F_{X' \times_X X'}$ all agree (up to isomorphism).

$$\xi'_{ij} \simeq \xi'_{ji} \qquad\qquad \xi' \qquad\qquad\qquad \xi$$

$$X' \times_X X' \overset{p_1}{\underset{p_2}{\rightrightarrows}} X' \xrightarrow{\qquad\alpha\qquad} X$$

$$(U_i \cap U_j) \qquad\qquad (U_i) \qquad\qquad\qquad (U)$$

Condition (B) says that in that case the objects in the family can be glued together along α, so as to be essentially equivalent to an object in F_X:

$$\xi' \simeq \alpha^*(\xi) \text{ for some } \xi$$

Reconstruction of the link with AST

An arrow $\alpha : X' \to X$ is a **strict descent morphism** if it is a strict F-descent morphism for $F = \ C^{\to} \xrightarrow{\text{cod}} C$.

[Grothendieck 1959, Proposition 2.1]
If C has finite products and pullbacks, then the strict descent morphisms in C are exactly the (universal strict) epimorphisms in C.

[Reformulation of Joyal-Moerdijk's Axiome 3]
Any epimorphism in C is a strict descent morphism for the sub-fibration $p_S : S \to C$ of the codomain fibration cod : $C^{\to} \to C$.

Proposition: A subcategory S of $C^{\to}$ satisfies the Axioms 1, 2 and 3 of a collection of small maps (in the sense of AST) **iff** p_S satisfies Grothendieck's property.

Reinterpretation of AST's first three axioms

- Axiom 1 says that all isomorphisms in C belong to S, which implies that, for any object A of C, 1_A belongs to S, and thus that there is at least one object (arrow) in S above each object of C. So the category S can really be said to be "above" C.
- Axiom 2 then says that any pullback of any arrow in S belongs itself to S, which ensures that the restriction to S of the codomain fibration $\mathrm{cod} : C^{\rightarrow} \rightarrow C$ is itself a fibration, in other words that p_S is a sub-fibration of $C^{\rightarrow} \rightarrow C$. (This is more than just saying that S is a subcategory of $C^{\rightarrow}$.)
- Axiom 3 finally says **that Grothendieck's result about cod** $: C^{\rightarrow} \rightarrow C$ **remains true about** $p_S : S \rightarrow C$.
  (The fact that C is a Heyting pretopos guarantees that any epimorphism in C is a universal strict epimorphism.)

A class of small maps can be partially characterized as a category to which descent theory applies.

For further work

In the context of descent theory, the natural extension of a fibration is a *stack*.

Gabriele Vezzosi: If a fibration is a family of categorical objects which "pullback like bundles," a stack is a family of such objects which moreover "glue like bundles."

The notion of stack makes sense only on the condition that the base category C is endowed with a *Grothendieck topology*.

A *Grothendieck topology* J on a category C consists of a collection $J(X)$ of "covering families" $\{X_i \to X\}$ on X, for each object X of C, in such a way that the different $J(X)$'s are connected in a systematic way.

Given a base category C endowed with a Grothendieck topology J, a **stack over** C is a fibration over C such that any covering $\{X_i \to X\}$ in $J(X)$ amounts to a strict descent morphism.
Examples: the fibration $\mathrm{dom} : \mathrm{Cont} \to \mathrm{Top}$ is a stack, $\mathrm{cod} : C^{\to} \to C$ is a stack for a certain topology on C.

Let $\langle C, S \rangle$ a Heyting pretopos C with a class S of small maps. Since C is a pretopos, there is a natural Grothendieck topology $\mathcal{T}_c$ on C, called the *coherent topology*, which is generated by all the finite jointly epimorphic families of arrows.

Natural question: For which properties of S is $p_S : S \to C$ ensured to be a stack w.r.t. the site $\langle C, \mathcal{T}_c \rangle$?

Conclusion

From a set-theoretic viewpoint, any map is in fact a set: there is no arrow. The only arrows are the edges of the membership graph.

On the contrary, AST reinterprets set theory so as to turn it into an arrow-based theory (along the codomain fibration), rather than an object-based theory (which is typical of the category-theoretic point of view).

There is more:

AST = graft of the theory of fibrations onto ZFC

AST = categorical implementation of ZFC with a deep geometrical twist, and as such an example of combination between “foundational mathematics” and “mathematical practice.”

THANK YOU !