# Category theory and set theory: examples of their interaction

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My talk will be composed of three parts:

1. quick review of the basic ping-pong that opposes the set-theoretic and the category-theoretic foundational perspectives
2. first example of interaction between CT and ST: sketch theory
3. another example, that ilustrates a kind of graft of CT onto ST: algebraic set theory.

# BASIC PING-PONG

<u>Arguments against ST.</u>

ARG 1: The very notion of category ("collection" of objects, "collection" of arrows) presupposes the notion of set.

ANS: No. Otherwise, any formal language (based on some "collection" of symbols) would presuppose set theory. This would apply to the formal language of ZFC itself.

ST presupposes some background universe as much as CT does. That universe has nothing to do with a model of ZFC.

ARG 2: Most mathematical notions are defined in set-theoretic terms.

ANS: No. Thinking in those terms is but confusing a mathematical theory with its set models, on the basis of Gödel's completeness theorem. The theory of fields, for instance, has nothing to do as such with set theory.

A category $C$ is *concrete* in case there exists a faithful functor from $C$ to *Set*.
Fact: There are non-concrete categories, such as higher categories.

ARG 3: The category *Set* is pervasive throughout mathematics. The Yoneda lemma itself proves that any category, however non-concrete it is, can be embedded into a category of sheaves of sets.

ANS: True, but actually *Set* is brought up almost always as the universal category with one object and stable under colimits – characterized only up to categorical equivalence. only, which is a way to keep track of the generality of most mathematical constructions.

ARG 4: The objects and arrows that a category is made of are set-theoretic constructions. The category itself is but a superstructure built on top of those. Otherwise, how to explain where objects of a category come from?

ANS: Functorial semantics. Sketch theory is a nice illustration. A mathematical object such as a field can be introduced as a functor from a category presenting the field structure, to some realization category (usually *Set*, but not necessarily).

Arguments against ST.

ARG 1': If you fix a base topos $S$ and work with $S$-toposes (toposes above $S$), the set theory constituted by the internal language of $S$ becomes the standard set theory. So ZFC is but one particular case of set theory.

ARG 2': Tom Leinster, "Rethinking set theory": "The approach described here [= Lawvere, "An elementary theory of the category of sets"] is not a rival to set theory: it *is* set theory."

ARG 3': Working with classifying toposes allows one to get a better grasp of theories.

The duality or rivalry between set theory (ST) and category theory (CT) is historical and sociological, coming from mathematical practice.

Championing either ST or CT is just extrapolating a tradition, or a certain field within mathematics (analysis and logical semantics in the case of ST, algebraic geometry and modern algebraic topology in the case of CT).

“Foundations” are provided neither by ST nor by CT.

Still, the opposition between ST and CT maybe revives and extends the old opposition between the syntactic, axiomatic approach to logic, with its basic notion of provability (Frege, Peano, Russell-Whitehead) and the semantical, set-theoretic approach, with its basic notion of validity (Schröder, Löwenheim, Skolem).

Hilbert and Ackermann's *Principles of Mathematical Logic* sealed this opposition, to the advantage of the second one. Set theory prevailed as a logical framework upon the axiomatic one.

Affinity between CT and the syntactic approach: see the theory of monads, sketch theory, the theory of syntactic fibrations, the notion of classifying topos, among others. Lawvere's functorial semantics is a syntactic semantics, where the axiomatic (proof-theoretic) approach prevails.

(Model theory: one theory, several models. Categorical logic: one classifying model, several theories.)

Is there some Hilbert-Ackermann redux to expect to settle the dispute between ST and CT?

No. It is not anyway about confronting, but about combining.

Two specific examples of how CT cooperates with ST:

- Sketch theory = categorical tool to control and justify the central place given to *Set*.
- Algebraic set theory = categorical implementation of ZFC, with a geometrical twist.

# SKETCH THEORY

Sketches were introduced by Charles Ehresmann in the late sixties, in the field of algebraic and differential topology.

Sketch = category-theoretic diagram that represents a specific kind of structures (monoids, groups, fields, . . . ).

Sketch = dynamic diagram: the proof of a property of all monoids amounts to adding vertices and arrows to the original sketch of monoids.
This is a way of combining the representation of an object with the representation of a piece of reasoning based on the representation of that object.

Sketch theory = systematic way to turn many mathematical theories into a diagrammatic presentation.

*Cone*: bundle of outgoing arrows

$$\begin{tikzcd}
\cdots & a_i \arrow[rr, "f_{i,j}"] & & a_j & \cdots \\
& & a \arrow[ull] \arrow[ul, "h_i"'] \arrow[ur, "h_j"] \arrow[urr] & &
\end{tikzcd}$$

$$f_{i,j} \circ h_i = h_j \;\; .$$

*Cocone*: bundle of ingoing arrows

$$\begin{tikzcd}
& & b & & \\
\cdots \arrow[urr] & b_i \arrow[ur, "g_i"] \arrow[rr, "f'_{i,j}"] & & b_j \arrow[ul, "g_j"'] & \cdots \arrow[ull]
\end{tikzcd}$$

$$g_j \circ f'_{i,j} = g_i \;\; .$$

*Limit cone*: this is a cone $(f_i : a \to a_i)_{i \in I}$ that mediates any other cone $(g_i : b \to a_i)_{i \in I}$ over the same diagram (= existence of $h : b \to a$ such that $g_i = f_i \circ h$ for any $i \in I$.

*Limit cocone*: analogous.

Sketch $S_f$ of commutative field:

$$
\begin{tikzcd}
 & 1 \arrow[dr] & & & & \\
 & & 1 + F^* \arrow[rr, "\text{iso}"] & & F & F \times F \arrow[l, bend right, "+"'] \arrow[l, bend left, "\times"] . \\
F^* \arrow[r, "(-)^{-1}"] & F^* \arrow[ur] & & & &
\end{tikzcd}
$$

Sketch = tracing pattern, each realization of which traces in some given category $A$ (for example in $Set$).

Example of Lie groups:

$$
\begin{tikzcd}
S_g \arrow[r, "G"] \arrow[rr, dashed, bend right, "TG"'] & Man \arrow[r, "T"] & Man
\end{tikzcd}
$$

A *sketch* is a quadruple $S = \langle |S|, D, C, C' \rangle$, where

- $|S|$ is a graph,
- $D$ a collection of commutative diagrams in $|S|$ (called *distinguished diagrams*),
- $C$ a collection of cones (called *distinguished cones*) in $|S|$,
- and $C'$ a collection of cocones (called *distinguished cocones*) in $|S|$.

In case $C' =$, $S$ is said to be *projective*.

A *morphism of sketches*
$f : \langle |S_1|, D_1, C_1, C'_1 \rangle \to \langle |S_2|, D_2, C_2, C'_2 \rangle$ is simply a functor $f : |S_1| \to |S_2|$ such that

- $f \circ d_1 \in D_2$ for any $d_1 \in D_1$,
- $f \circ c_1 \in C_2$ for any $c_1 \in C_1$
- and $f \circ c'_1 \in C'_2$ for any $c'_1 \in C'_1$.

A *realization* of a given sketch $S$ is a functor from the underlying graph $|S|$ of $S$ into some category $A$ that turns

- every distinguished diagram of $D$ into a commutative diagram in $A$,
- every distinguished cone of $C$ into a limit cone in $A$
- and every distinguished cocone of $C'$ into a limit cocone in $A$.

In the case where the *realization category* $A$ is *Set*, a realization of a sketch is called a *model* of this sketch.

Functorial semantics: a field is nothing but a model of the sketch of fields; a group, a model of the sketch of groups, and so forth.

In the framework of sketches, a proof becomes itself a sketch that constitutes a specific enrichment of the original sketch of a theory.

Example given by L. Coppey, “Esquisses et types” (1992): the sketch of monoids.
Basic sketch $S_0$:

$$\begin{tikzcd} M^2 \arrow[r, bend left, "p_1"] \arrow[r, "k"] \arrow[r, bend right, "p_2"'] & M^1 & M^0 \arrow[l, "e"'] \end{tikzcd} .$$

($M^1$ = underlying set, $M^0$ = singleton, $e$ = distinguished element.)

This sketch works as a a first order multisorted language:

- sorts = objects of $S_0$ ;
- function symbols $f : X \to Y$ are arrows of $S_0$ ;
- $\forall x : X \;\; f'(f(x)) = g(x)$ iff $f'f = g$ in $S_0$.

Sketch $S_1$:

$$\begin{tikzcd}
M^3 \arrow[r, shift left, "k_1"] \arrow[r, shift right, "k_2"'] \arrow[r, bend left=40, "r_1"] \arrow[r, bend left=70, "r_2"] \arrow[rr, bend right=40, "q_1"] \arrow[rr, bend right=55, "q_2"] \arrow[rr, bend right=70, "q_3"'] & M^2 \arrow[r, "k"] \arrow[r, bend left=20, "p_1"] \arrow[r, bend right=30, "p_2"'] & M^1 \arrow[l, bend right=40, "v_1"'] \arrow[l, bend right=70, "v_2"'] \arrow[r, bend right=10, "u"'] & M^0 \arrow[l, bend right=10, "e"']
\end{tikzcd} .$$

The $p_i$'s are the natural projections, which is expressed by the choice of $M^1 \xleftarrow{p_1} M^2 \xrightarrow{p_2} M^1$ as a distinguished cone. Same thing for the $q_i$'s.

Added arrows in the transition from $S_0$ to $S_1$:

$(r_1)$ $p_1 r_1 = q_1, p_2 r_1 = q_2$ (that is, $r_1 : (x, y, z) \mapsto (x, y)$);
$(r_2)$ $p_1 r_2 = q_2, p_2 r_2 = q_3$ (that is, $r_2 : (x, y, z) \mapsto (y, z)$);
$(k_1)$ $p_1 k_1 = kr_1, p_2 k_1 = q_3$ (that is, $k_1 : (x, y, z) \mapsto (xy, z)$);
$(k_2)$ $p_1 k_2 = q_1, p_2 k_2 = kr_2$ (that is, $k_2 : (x, y, z) \mapsto (x, yz)$);
$(v_1)$ $p_1 v_1 = 1_{M^1}, p_2 v_1 = eu$ (that is, $v_1 : x \mapsto (x, e)$);
$(v_2)$ $p_1 v_2 = eu, p_2 v_2 = 1_{M^1}$ (that is, $v_2 : x \mapsto (e, x)$);
$(u)$ $u$ is the only possible map from $M^1$ to $M^0$.

Added axioms:

$kv_1 = kv_2 = 1_{M^1}$;
$kk_1 = kk_2$.

The equalities satisfied by the $r_i$'s and the $k_i$'s, as well as the added axioms, amount to commutativity conditions which are expressed by putting forward distinguished diagrams or distinguished cones.

For example, stating that $kk_1 = kk_2$ amounts to picking

$M^3 \underset{k_2}{\overset{k_1}{\rightrightarrows}} M^2 \xrightarrow{k} M^1$ as a distinguished diagram.

Given a monoid $(M, .)$, the 3-associativity of $.$ entails its 4-associativity:

$$(x(yz))t = x(y(zt)) \quad (①).$$

*Expressing* ① requires a new sketch $S_1'$ which supplements $S_1$ with a new vertex $M^4$, new arrows $u_i : M^4 \to M^3$, $t_j : M^4 \to M^2$, $s_l : M^4 \to M^1$, and new equalities about these arrows.

$$\begin{tikzcd}
M^4 \arrow[r, "u_i"] \arrow[rr, bend left=40, "t_j"] \arrow[rrr, bend left=60, "s_l"] & M^3 \arrow[r, shift left, "k_1"] \arrow[r, shift right, "k_2"'] \arrow[r, bend left=60, "r_1"] \arrow[r, bend left=80, "r_2"] \arrow[rr, bend right=40, "q_1"] \arrow[rr, bend right=60, "q_2"] \arrow[rr, bend right=80, "q_3"'] & M^2 \arrow[r, "k"] \arrow[r, bend left=30, "p_1"] \arrow[r, bend right=30, "p_2"'] & M^1 \arrow[l, bend right=60, "v_1"'] \arrow[l, bend right=80, "v_2"'] \arrow[r, bend right=10, "u"'] & M^0 \arrow[l, bend right=10, "e"']
\end{tikzcd} .$$

Again, the equalities are in fact laid down by mentioning distinguished diagrams and cones.

*Establishing* ① requires the addition of some new arrows $k'_i : M^4 \to M^3$ ($i = 1, 2, 3$) defined by equalities in such a way that $k'_1$, for example, represents the map $(x, y, z, t) \mapsto (xy, z, t)$.

In the new sketch $S''_1$ that is obtained, ① is $k(k_1 k'_2) = k(k_2 k'_3)$ and can be proved, provided the realization category has finite products.
For any category $A$ with finite products, any realization of $S_1$ in $A$ can be extended into a model of $S''_1$ in $A$.

To sum up: 4-associativity results from 3-associativity through a progression $S_1 \to S'_1 \to S''_1$ consisting in two successive sketch inclusions (sketch morphisms).

A proof becomes a specific completion of a given sketch: the addition of the vertices and arrows that make possible the proof of a given theorem.

$S_1''$ = explicit presentation of the proof of ①, in the sense that the sketch $S_1''$ shows exactly what has to be added to $S_1$ in order for ① to actually hold. The validity of ① becomes an obvious part of $S_1''$.
Any *realization* of $S_1''$ will obviously validate ① (4-associativity), because $S_1''$ explicitly prescribes it.

As Peter Freyd puts it, the task of the mathematician is "to make trivially trivial what is trivial." Here $S_1''$ is what discharges this task in the case of ①: $S_1''$ can be described as the *completion* of $S_1$ w.r.t. ①.

Sketches have the same expressive power as infinitary first-order logic.

But compare sketches to formal theories construed as deductively closed sets of sentences.
In the latter case, there is no way to pinpoint the step at which some specific theorem is obtained.
On the contrary, the transition from $S_1$ to $S_1''$ (sketch enrichment) is exactly adjusted to the proof of a peculiar theorem.

Sketches = diagrammatic way of pinpointing a local proof-theoretic fact.

Sketch theory = diagrammatic theory of mathematical theories turned into diagrams.

The *models* of $S_1$ and $S_1''$ are the same, despite the fact that nothing in $S_1$ allows one to see ①.
This is because the category *Set* automatically adds to any model of $S_1$ what is needed to get a model of $S_1''$. This is due to a property of *Set*, namely having finite products. This does not always happen, as we will see.

A monoid can also been defined as a set $M$ endowed with an associative binary law $k(x, y)$ such that:

$$\exists e \in M\, \forall x \in M\; k(x, e) = k(e, x) = x \quad (②).$$

Fact: ② is sketchable, by a sketch $S_2$.

Fact: the models (in *Set*) of $S_2$ coincide with the models of $S_1$.

Fact: Given a $K$-vector space $M$, there is only one way of turning $M$ into a realization of $S_1$ ($k(x, y) = x + y$ and $e = \vec{0}$).

Fact: For any couple $(\lambda, \mu)$ of commuting projectors on $M$, $k(x, y) := \lambda(x) + \mu(y)$ is a binary law on $M$ turning $M$ into a realization of $S_2$.

Conclusion: There are realizations of $S_2$ in $K$-Vect which are not realizations of $S_1$. (Many possible choices for the complement of a linear subspace.)

Conclusion: $S_1$ and $S_2$ have the same realizations in *Set*, but not in $K$-Vect.

This is because *Set* has a feature that $K$-Vect does not have: the uniqueness of complements.

# General analysis of what happens

Any sketch morphism $f : S \to S'$ gives rise to a functor $U_f = - \circ f : Set^{S'} \to Set^S$:

$$\begin{tikzcd} S \arrow[rr, "f"] \arrow[dr, dashed, "U_f(Y)=Y\circ f"'] & & S' \arrow[dl, "Y"] \\ & Set & \end{tikzcd}$$

$U_f$ = "forgetful functor" associated to $f$.

A model $X$ of $S$ is a model of $S'$ iff $X : S \to Set$ factorizes through $f$.

Ehresmann 1967: For any morphism $f$ of projective sketches, $U_f$ has a left adjoint $F_f$.

Then, for any model $X : S \to Set$ of $S$,
$T_f(X) := U_f(F_f(X))$ = "$f$-structure freely generated by $X$".

Small categories are themselves projectively sketchable. There is a (projective) sketch of all small categories:

$$
\begin{tikzcd}[column sep=huge]
C_0 \arrow[r, "x \mapsto 1_x"] & C_1 \arrow[l, bend right=50, "\mathrm{cod}"'] \arrow[l, bend left=50, "\mathrm{dom}"] & C_2 \arrow[l, bend right=20] \arrow[l, bend left=20]
\end{tikzcd}\ ,
$$

($C_0$ = collection of objects, $C_1$ = collection of arrows,
$C_2 = \{(f, g) \in C_1 \times C_1 : \mathrm{cod}(f) = \mathrm{dom}(g)\}$).
(I simplify cardinality constraints.)

Small cones as well as small cocones can be sketched.

Small sketches are themselves projectively sketchable. Let $\sigma$ be the sketch of all small sketches. A small sketch $S$ is the same thing as a model $F_S : \sigma \to SET$ of $\sigma$.

Any projectively sketchable property of sketches corresponds to a morphism $p : \sigma \to \sigma'$ of projective sketches, **to which the previous result applies**.

$$\begin{tikzcd} \sigma \arrow[rr, "p"] \arrow[dr, dashed, "S"'] & & \sigma' \arrow[dl] \\ & SET & \end{tikzcd}$$

A sketch $S$ (a model of $\sigma$) is a $p$-sketch (a model of $\sigma'$) iff $S$ factorizes through $p$.

Each sketch $S$ gives rise to its $p$-type $T_p(S)$, the smallest extension of $S$ having property $p$.
A sketch $S$ is a $p$-sketch iff $S \simeq T_p(S)$.

For any sketch $T$ having property $p$:

$$\mathrm{Hom}_{SET^S}(S, U_p(T)) \simeq \mathrm{Hom}_{SET^{S'}}(F_p(S), T)$$

Since $U_f$ is full and faithful:

$$\mathrm{Hom}_{SET^{S'}}(F_p(S), T) \simeq \mathrm{Hom}_{SET^S}(U_pF_p(S), U_p(T))$$

Therefore:

$$\mathrm{Hom}_{SET^S}(S, U_p(T)) \simeq \mathrm{Hom}_{SET^S}(T_p(S), U_p(T))$$

Identifying $T$ and $U_p(T)$:

$$\mathrm{Hom}_{SET^S}(S, T) \simeq \mathrm{Hom}_{SET^S}(T_p(S), T)$$

In particular, if the category *Set* (viewed as a sketch) has property $p$, one can take $T = Set$, which means that $S$ and $T_p(S)$ have the same models.

Conclusion: Let $S$ be any sketch of a certain kind of structures. Then:

$$Set^S \simeq Set^{T_p(S)} \text{ whenever } Set \simeq T_p(Set)$$

Remark: in "$Set^S \simeq Set^{T_p(S)}$", *Set* is taken as a category (and thus as a sketch); in "$Set \simeq T_p(Set)$", *Set* is taken as model of $\sigma$.

Imagine one is interested in a certain possible theorem $\theta$ about a certain kind of structures, of sketch $S$.
Suppose $\theta$ corresponds to a projectively sketchable property.
Let $S \xrightarrow{\theta} S_\theta$ the corresponding sketch morphism.

The obtaining of $A^S \simeq A^{S_\theta}$ can then be seen as a property $p_\theta$ of $A$, sketchable through a sketch morphism $\sigma \xrightarrow{p_\theta} \sigma_\theta$ .

For any category $A$, $A^S \simeq A^{S_\theta}$ iff $A \simeq T_{p_\theta}(A)$.

In particular, $\theta$ holds in all models of $S$ iff $Set^S \simeq Set^{S_\theta}$ iff $Set \simeq T_{p_\theta}(Set)$.

This is exactly what happened with $S_1$ and $S_1''$ and with $S_1$ and $S_2$.

One has: $S_1 \xrightarrow{①} S_1''$ and $\sigma \xrightarrow{p_{①}} \sigma_{①}$, where $\sigma_{①}$ sketches the categories having 4-products.

One has $S_2 \xrightarrow{②} S_1$ and $\sigma \xrightarrow{p_{②}} \sigma_{②}$, where $\sigma_{②}$ sketches the categories having unique complements.

$$Set = T_{p_{②}}(K\text{-Vect}),$$

where $p_{②}$ is the property confusing $S_1$ and $S_2$ (the minimal property $p$ such that $p$-realizations of $S_1$ and $p$-realizations of $S_2$ coincide).

When we interpret the sketch $S_1$ of monoids by looking to *models* of this sketch, the fact that we are dealing with limit cones and working within *Set adds* arrows to the original schema of $S_1$.

Indeed, to any cone in $S_1$ will correspond in *Set* a limit cone *c* with all the cones sharing the same basis as *c* and all the mediating arrows $\phi$ induced between such cones and *c*.

This addition forces 4-associativity (①) in any model of $S_1$ on top of the equalities explicitly true in $S_1$.

The question is then: for which minimal property $p$ of a category $A$ are there enough arrows in $A$ so that ① is true in any $A$-realization of $S_1$? The answer is: $p$ is the property $p_{①}$ of having 4-products.

$T_{p_{①}}(S_1)$ is the minimal extension of $S_1$ having that property.

$$\begin{tikzcd} \sigma \arrow[rr, "p_{①}"] \arrow[dr, "S_1"'] & & \sigma_{①} \arrow[dl, "F_{p_{①}}(S_1)"] \\ & SET & \end{tikzcd}$$

Quite generally, the choice of *Set* for the realization of a sketch $S$ in the context of the proof of a theorem $\theta$ hinges on whether $Set = T_{p_\theta}(S)$.

On the condition that $T_{p_\theta}(S)$ is a limit sketch (i.e., a sketch whose cones are all limit ones), the canonical inclusion $S \to T_{p_\theta}(S)$ can be seen as a semantic realization of $S$, in addition to being a sketch morphism (a proof-theoretic enrichment of $S$).

$T_{p_\theta}(S)$ is a category in which $S$ can be adequately realized from the point of view of $\theta$, because $T_{p_\theta}(S)$ is complete with respect to the relevant limits.

$T_{p_\theta}(S)$ = the universal realization category $A$ such that any realization of $S$ in $A$ satisfies $\theta$.
$T_{p_\theta}(S)$ = the good semantical environment as soon as one is interested in the truth of $\theta$ in the context of the theory sketched by $S$.

*Set* is not necessarily the best semantics: everything depends on the theorem $\theta$ (or bunch of theorems) at stake.

In some cases, *Set* turns out to be richer than $T_{p_\theta}(S)$, which retains only what is strictly necessary to make $\theta$ true in all realizations.

In some other cases, *Set* may not be rich enough.

In many cases, *Set* proves to be the most accurate realization category possible.

*Sketch theory carries out a principled category-theoretic arbitration between the category of sets and other possible realization categories.*

Remark: this is more constructive than a mere alternative proposal to replace set theory.

# ALGEBRAIC SET THEORY (AST)

AST = "graft" of the theory of fibrations (coming from CT) onto ST (ZFC).

Original motivation: the internal language of a topos allows only for bounded quantification ($\forall x : X \varphi(x)$).

Two different approaches : a logical one (Simpson, Awodey), a geometrical one (Joyal-Moerdijk).

Joyal & Moerdijk: generalizing models of ZF ("free ZF-algebras").

Awodey & co: general program of completeness results between axiomatizations of set theory and collections of "categories of classes".

Let $C$ be a Heyting pretopos.

The idea is to characterize a class $S$ of "small maps" in $C$, an arrow being a small map if all its fibers have a set-like size.

Then, an object $X$ of $C$ is said to be *small* if $X \to 1$ is a small arrow.

The axioms for the class $S$:

1. Any isomorphism of $C$ "belongs" to $S$ and $S$ is closed under composition.
   This axiom corresponds to the fact that the union of a small family of small sets is a small set.
2. Stability under change base: for any pullback

$$\begin{tikzcd} Y' \arrow[r] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ X' \arrow[r, "p"'] & X \end{tikzcd},$$

   if $f$ belongs to $S$, so does $g$.
3. "Descent": for any pullback along an epimorphic arrow $p$

$$\begin{tikzcd} Y' \arrow[r] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ X' \arrow[r, two heads, "p"'] & X \end{tikzcd},$$

   if $g$ belongs to $S$, so does $f$.

4. The arrows $0 \to 1$ and $1 + 1 \to 1$ belong to $S$.
   This axiom ensures that the empty set (initial object) is small and that any finite set is small.
5. If two arrows $f : Y \to X$ and $f' : Y' \to X'$ belong to $S$, then so does their sum $f + f' : Y + Y' \to X + X'$.
   This axiom ensures that the disjoint union of two small sets (objets) is a small set.
6. In any commutative diagram

$$\begin{tikzcd} Z \arrow[rr, two heads, "p"] \arrow[dr, "g"'] & & Y \arrow[dl, "f"] \\ & B & \end{tikzcd},$$

   if $g$ belong to $S$, then so does $f$.

7. Collection axiom: For any epimorphic arrow $p : Y \twoheadrightarrow X$ and any arrow $f : X \longrightarrow A$ that belongs to $S$, there exists a quasi-pullback:

$$
\begin{tikzcd}
Z \arrow[rr] \arrow[dr, two heads, "\text{epi}"] \arrow[dd, "g"'] & & Y \arrow[r, two heads, "p"] & X \arrow[dd, "f"] \\
& X \times_A B \arrow[urr, dashed] \arrow[dl, dashed] & & \\
B \arrow[rrr, two heads, "\text{épi}"'] & & & A
\end{tikzcd}
$$

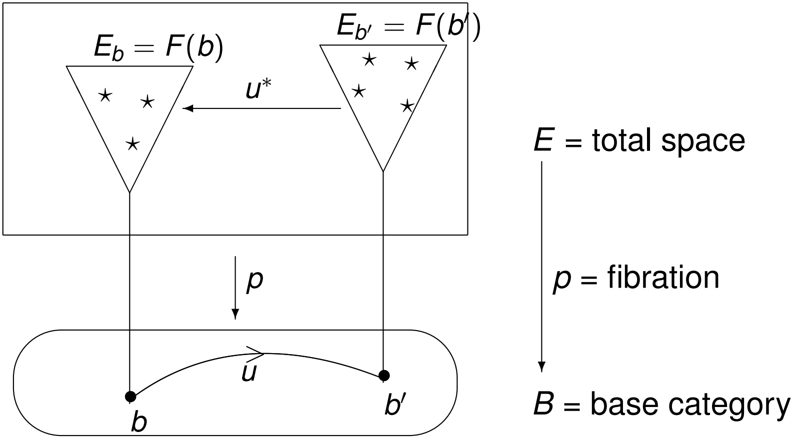
with $g$ in $S$.
This axiom corresponds to the following set-theoretic principle:
If $\forall y \in x\ \exists z \phi(y, z)$, then $\exists w\ \forall y \in x\ \exists z \in w\ \phi(y, z)$ and $\forall z \in w\ \exists y \in x\ \phi(y, z)$.

8. Any arrow $f : Y \to X$ in $S$ is exponentiable : $- \times f : (Z \to X) \mapsto (Z \times_X Y \to X)$ admits of a right adjoint. This axiom ensures that if $X$ is small, then $Z^X$ is small, for any object $Z$.
9. $S$ contains a universal arrow $\pi : E \to U$ : for any arrow $f : Y \to X$ in $S$, there exists a double pullback:

$$\begin{tikzcd} Y \arrow[d, "f"'] & Y' \arrow[l] \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\ X & X' \arrow[l, two heads, "\text{epi}"] \arrow[r] & U. \end{tikzcd}$$

   In other words, any arrow in $S$ is “locally” a pullback of $\pi$. I will get back to the term “locally”. The object $U$ is the small objects classifier.

Fibration (fibred category) = generalization in CT of the notion of surjective map.




A fibration in general

Example: $B$ = category *Top* of topological spaces, $E$ = category *Fib* of fiber bundles.

Example: object in *Top* = manifold $M$, $f : N \to M$ in *Top*, object in $Fib_M$ = tangent bundle $TM$:
$f^*(TM)$ is defined by pullback. More on that soon.

Extra-ingredient supplied by a fibration: the base category is endowed with some structure and a systematic correlation must exist between connections ($u$) between any two objects in the base and certain connections ($u^*$) between the corresponding fibers.

A fibration is a surjective functor for which context change behaves well.

## Reconstruction of the link with AST

Given a category $C$, let $C^{\to}$ be the category of all arrows in $C$. The codomain functor $\mathrm{cod} : C^{\to} \to C$ is a fibration.

The fiber above each $X \in \mathrm{Ob}\, C$ is the category $C/X$ composed of all arrows in $C$ with codomain $X$.

Any arrow $\alpha : X' \to X$ in $C$ gives rise (by pullback) to a base change functor $\alpha^* : C/X \to C/X'$.

$$
\begin{tikzcd}
T \times X' \arrow[r, dashed] \arrow[d, dashed, "\alpha^*(f)"'] & T \arrow[d, "f"] \\
X' \arrow[r, "\alpha"'] & S
\end{tikzcd}
\quad \text{becomes:} \quad
\begin{tikzcd}
\alpha^*(f) \in C/X' & f \in C/X \arrow[l, dashed, mapsto, "\alpha^*"'] \\
X' \arrow[r, "\alpha"'] & X
\end{tikzcd}
$$

*The class $S$ of small maps is thought of as a sub-fibration $S \to C$ of the codomain fibration $\mathrm{cod} : C^{\to} \to C$.*

Descent theory = abstract framework geared to describe glueing processes.

Example. Let $(X_1, X_2)$ be a covering of a topological space $X$ and suppose:

- that a fiber bundle $V_i$ is defined above $X_i$ for each $i = 1, 2$
- that there is an isomorphism $V_1|_{X_1 \cap X_2} \simeq V_2|_{X_1 \cap X_2}$

Then, it is possible to glue $V_1$ and $V_2$ together so as to obtain a fiber bundle above the whole space $X$.

Central example of fiber bundles, which are a central example of a *fibred category*.
Hence the link of descent theory with fibrations.

Grothendieck, “Technique de descente et théorèmes d’existence en géométrie algébrique”(1959).

Let $F$ be a fibred category with base category $C$. For any arrow $f : X \to Y$ in $C$ and any $\xi \in F_Y$,

$$\begin{tikzcd} f^*(\xi) = \xi \times_Y X \arrow[r, dashed] \arrow[d, dashed] & \xi \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

defines a functor $f^* : F_Y \to F_X$.

Let $\xi' \in F_{X'}$ and let $\beta_1, \beta_2 : X'' \to X'$ two arrows in $C$.

A *glueing datum ("donnée de recollement") on* $\xi'$ *w.r.t. the pair* $(\beta_1, \beta_2)$ is an isomorphism $\beta_1^*(\xi') \simeq \beta_2^*(\xi')$.

A *descent datum ("donnée de descente") on* $\xi'$ *w.r.t. the arrow* $\alpha : X' \to X$ is a glueing datum on $\xi'$ w.r.t. the following pair $(p_1, p_2)$ of canonical projections:

$$\begin{tikzcd} X' \times_X X' \arrow[r, dashed, "p_2"] \arrow[d, dashed, "p_1"'] & X' \arrow[d, "\alpha"] \\ X' \arrow[r, "\alpha"'] & X \end{tikzcd}$$

Explanation: Example of $(X_1, X_2)$. For $X' = X_1 \coprod X_2$, there is an obvious surjective map $\alpha : Y \to X$.

$$\begin{tikzcd} X' \times_X X' \arrow[r, dashed, "p_2"] \arrow[d, dashed, "p_1"'] & X' \arrow[d, "\alpha"] \\ X' \arrow[r, "\alpha"'] & X \end{tikzcd}$$

Here $X' \times_X X' \simeq X_1 \coprod X_2 \coprod (X_1 \cap X_2)$.

So the required isomorphism $p_1^*(\xi') \simeq p_2^*(\xi')$ means in particular that the component of $\xi'$ on $X_1$ and the component of $\xi'$ on $X_2$ match up on $X_1 \cap X_2$ and thus have a chance to be glued together.

An arrow $\alpha : X' \to X$ in $C$ is an *$F$-descent morphism* ("morphisme de $F$-descente") if giving an object of $F_X$ amounts to giving an object of $F_{X'}$ equipped with a descent datum w.r.t. $\alpha$:

- $\alpha p_1 = \alpha p_2$
- for any two objects $\xi, \eta$ of $F_X$, $\alpha^*$ induces a 1-1 map from $\mathrm{Hom}_{F_X}(\xi, \eta)$ to the subset of $\mathrm{Hom}_{F_{X'}}(\alpha^*(\xi), \alpha^*(\eta))$ composed of all arrows $v : \alpha^*(\xi) \to \alpha^*(\eta)$ such that $p_1^*(v) = p_2^*(v)$.

Explanation: Example of $X' = X_1 \coprod X_2$ again.

$$\begin{tikzcd} X' \times_X X' \arrow[r, dashed, "p_2"] \arrow[d, dashed, "p_1"'] & X' \arrow[d, "\alpha"] \\ X' \arrow[r, "\alpha"'] & X \end{tikzcd}$$

Let $\xi, \eta \in F_X$
$\mathrm{Hom}_{F_X}(\xi, \eta) \longleftrightarrow \{v \in \mathrm{Hom}_{F_{X'}}(\alpha^*(\xi), \alpha^*(\eta))/p_1^*(v) = p_2^*(v)\}$
The right-to-left direction is the meaningful one.
$v : \alpha^*(\xi) \to \alpha^*(\eta)$, hence $p_1^*(v) : p_1^*\alpha^*(\xi) \to p_1^*\alpha^*(\eta)$

$$\begin{tikzcd} p_1^*(v): \arrow[d, dashed, equal] & (\alpha p_1)^*(\xi) \arrow[r] \arrow[d, equal] & (\alpha p_1)^*(\eta) \arrow[d, equal] \\ p_2^*(v): & (\alpha p_2)^*(\xi) \arrow[r] & (\alpha p_2)^*(\eta) \end{tikzcd}$$

So the condition on $v$ implies that the $X_1$-component of $v$ and the $X_2$-component of $v$ agree. The condition says that if one have that kind of glueing device for any $v$, then giving an object in $F_{X'}$ actually amounts to giving an object in $F_X$.

An arrow $\alpha : X' \to X$ is a *descent morphism* (“morphisme de descente”) if it is an $F$-descent morphism for $F = \ C^{\to} \xrightarrow{\text{cod}} C$ .

$$
\begin{tikzcd}
\alpha^*(X) \arrow[r, "{\alpha^*(u),v}"] \arrow[d, "\alpha^*(\xi)"'] & \alpha^*(Y) \arrow[dl, "\alpha^*(\eta)"] \\
X' &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "\xi"'] & Y \arrow[dl, "\eta"] \\
S &
\end{tikzcd}
$$

$$
\begin{tikzcd}
X' \times_X X' \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & S' \arrow[r, "\alpha"] & X
\end{tikzcd}
$$

[Grothendieck 1959, Proposition 2.1]
If $C$ has finite products and pullbacks, then the descent morphisms in $C$ are exactly the (universal strict) epimorphisms in $C$.

[Reformulation of Joyal-Moerdijk's Axiome 3]
Any epimorphism in $C$ is a $p$-descent morphism for the sub-fibration $p_S : S \to C$ of the codomain fibration $\mathrm{cod} : C^{\to} \to C$.

**Proposition**: A subcategory $S$ of $C^{\to}$ is a class of small maps iff $p_S$ satisfies Grothendieck's property.

Better understanding of AST's first three axioms.

- The fact that $C$ is a Heyting pretopos guarantees that any epimorphism in $C$ is a universal strict epimorphism.
- Axiom 1 says that all isomorphisms in $C$ belong to $S$, which implies that, for any object $A$ of $C$, $1_A$ belongs to $S$, and thus that there is at least one object (arrow) in $S$ above each object of $C$. So the category $S$ can really be said to be "above" $C$.
- Axiom 2 then says that any pullback of any arrow in $S$ belongs itself to $S$, which ensures that the restriction to $S$ of the codomain fibration $\mathrm{cod} : C^{\to} \to C$ is itself a fibration, in other words that $p_S$ is a sub-fibration of $C^{\to} \to C$. (This is more than just saying that $S$ is a subcategory of $C^{\to}$.)
- Axiom 3 finally says that Grothendieck's result remains true about $p_S : S \to C$.

*A class of small maps can be partially characterized as a category to which descent theory applies.*

From a set-theoretic viewpoint, any map is in fact a set: there is no arrow.
The only arrows are the edges of the membership graph.

AST reinterprets set theory so as to turn it into an arrow-based theory (along the codomain fibration), rather than an object-based theory – this is typical of the category-theoretic point of view.
AST = graft of the theory of fibrations onto ZFC, which is more fruitful than the standard rivalry between ST and CT.

Application: Presentation of Tarski's semantics in a fibrational setting.

THANK YOU !