Indiscernibility in mathematics

Brice Halimi

University Paris Ouest (France)

PLAN

My talk will pursue two goals.

- First, I will show that combinatorics provides good examples of what a *mis*understanding in mathematics can be, and may thus cast some light on mathematical understanding and mathematical objectivity.
- This point comes close to the second goal of my talk, which is to reconsider the "identity problem" faced by the structuralist interpretation of mathematics.

Ante rem structuralism

According to *ante rem* structuralism, mathematics studies structures, conceived of as certain configurations of pure relata.

Their constituents do not have any individuality, since each of them is entirely characterized (individuated/identified) by the relationships it has with all other constituents.

In other words, a mathematical object is but an item within a structure, where it can have *relational* properties only.

The essence of a natural number is its relations to other natural numbers. The subject matter of arithmetic is a single abstract structure, the pattern common to any infinite collection of objects that has a successor relation with a unique initial object and satisfies the (second-order) induction principle. The number 2, for example, is no more and no less than the second position in the natural-number structure; 6 is the sixth position. Neither of them has any independence from the structure in which they are positions, and as places in this structure, neither number is independent of the other. (Shapiro, Philosophy of mathematics. Structure and Ontology, 1997, p. 72)

Structuralism is a strong form of platonism about mathematics, since it claims structures to exist, and to exist prior both to their places and to any exemplification that they may have in the world.

A given collection of individual objects, with relations between them, is what Shapiro calls a system.

A structure is "the abstract form of a system."

However, a structure is not the mere result of its abstraction from the diverse systems that instantiate it.

Indeed, Shapiro distinguishes between the "*places-are-offices* perspective" and the "*places-are-objects* perspective."

- In the first one, objects are what fill the places of structures. This corresponds to *in re* structuralism: places are but offices, and do not exist apart from their particular instances. Each instanciation of the structure is its realization.
- In the second perspective, places themselves are taken to be full-fledged objects in their own right, and cannot be reduced to mere façons de parler. Each instanciation of the structure is its emanation.

Shapiro's structuralism corresponds to the second option.

Salient feature of *ante rem* structures: A place in a structure is entirely determined by a bundle of relations which is constitutive of the structure and which connects that given place to all the other places of the structure, by virtue of the very nature of this structure.

As a result, *ante rem* structuralism faces one major problem: the problem of intra-structural identity.

The Identity Problem

The problem of intra-structural identity is the problem as to whether two distinct objects of the same structure can have exactly the same structural properties and yet be distinct.

This is the "identity problem," for short, as it has been brought up by Jukka Keränen in [Keränen, 2001]:

There are structures in which two distinct places are still structurally indistinguishable (despite being distinct).

Very elementary examples of that situation are the points of the geometric plane, or the conjugate complex numbers (i.e., $a + ib$ and $a - ib$).

Keränen focuses on i and $-i$ in the structure $\mathbb{C}$ of complex numbers.
Any formula in the language of $\mathbb{C}$ that is true of i is equally true of $-i$.

How to conceive of two distinct yet structurally indiscernible places of the same structure?

Any structure which admits of a nontrivial automorphism will raise the same problem.

In Keränen's view, any theory of a certain kind of objects has to provide one with individuation criteria for those objects.

The *ante rem* structuralist, however, cannot but individualize a place within a structure by the collection of relations that this place has with all the other places of the same structure.

As a consequence, any structure endowed with a nontrivial automorphism is a violation of the only identity criteria that are available to the *ante rem* structuralist, who is thus unable to account for such basic mathematical domains as Euclidean geometry or complex analysis.

In answer to Keränen's point, Shapiro points out that it is always possible to differentiate two given distinct objects. To be specific, the pair $\langle i, -i \rangle$ satisfies the formula $x + y = 0$, whereas $\langle i, i \rangle$. does not. This suffices to vindicate the fact that i and $-i$ are distinct.

Shapiro's strategy thus consists in substituting the task of differentiating two given objects (what Keränen calls a *condition of identity*] for a given object for the task of individuating any object in general (what Keränen calls an *account of identity*)

There is an ambiguity in Keränen's formulation. It comes from the back and forth between a system S and its underlying structure **S**.

> *[According to ante rem structuralism] each place of* **S** *is individuated by the 'purely structural' relational properties its occupants have to the occupants of the other places of* **S***. Thus, whenever two distinct elements in S that have the same intra-systemic relational properties, [...] they occupy the same place of* **S***. While this conclusion should seem prima facie plausible, in many cases it leads into absurdities. ([Keränen, 2006], p. 147-148)*

Keränen assumes that different places in a system S could correspond to a single place in the corresponding structure **S**.

But this assumption, according to Shapiro, is wrong: The non-coincidence of two places in a structure is shown by the non-coincidence of the two items that occupy those places in a given system. This constitutes an irreducible datum. As Shapiro puts it, "we know what identity is:"

Two items being relationally indiscernibles in a system S which instantiates a structure $\mathbf{S}$ *only* means that the system S' obtained by swapping one for the other is still a system that instantiates $\mathbf{S}$: The two items do not need to be identified. Shapiro is right in that.

On the other hand, Keränen is equally right in that the realist structuralist has no choice but to acknowledge that two relationally indiscernibles places within the one and same structure are identical.

More than a case study: Combinatorics

There is a privileged domain in mathematics which calls for its consideration —a domain where permutations and symmetries are brought to the fore, namely combinatorics.

Studying the very basic framework of combinatorics should give one the insight needed to understand how automorphisms and symmetries work, and thus how to reconsider the Keränen *vs* Shapiro controversy.

This will allow us to develop the second thread I would like to present: misunderstanding in mathematics.

Suppose that three items *a*, *b*, *c* are given, and let's consider the permutation that swaps *a* and *b* while leaving *c* unchanged.
This is a very elementary mathematical scenario. But is it so simple a thing to grasp?

The set composed of *a*, *b* and *c* is not altered by the permutation under consideration: This permutation is, so to speak, a mere fancy of the mind.

Yet we cannot help but picture *a*, *b* and *c* spatially, assigning to those objects different respective positions.
Doing that, we really could have put originally *a* at the position that is actually occupied by *b*, and put originally *b* at *a*'s actual position.

So how to conceive of the permutation, since it is supposed to be carried it out with respect to a setting which is itself defined only up to any arbitrary permutation of the original positions of *a*, *b* and *c* ?

It is really something difficult to get one's head around.

A permutation is a one-to-one mapping of a set to itself. One usually writes a permutation by using schematic letters. For instance, the permutation on a three-object set which exchanges the two first ones and leaves the third one untouched is usually written:

$$\begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}.$$

Now, it is obvious that the choice of $\{a, b, c\}$ instead, say, $\{\alpha, \beta, \gamma\}$ or $\{a_1, a_2, a_3\}$, is completely immaterial.

This does not mean, however, that one is considering the set $\{a, b, c\}$ up to a permutation "replacing" a with α, b with β, and c with γ. Indeed, keeping track of permutations precisely presupposes that letters have been settled once and for all.

The letters a, b and c are variable parameters to the extent that they are arbitrary, but —of course— they are not variable in the sense of the variation that they make it possible to represent.

Threefold arbitrariness attached to any representation of a permutation:

- the arbitrariness of the underlying set:

$$\begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix} \equiv \begin{pmatrix} \alpha & \beta & \gamma \\ \beta & \alpha & \gamma \end{pmatrix}.$$

- the arbitrariness of the original arrangement of the members:

$$\begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix} = \begin{pmatrix} b & a & c \\ a & b & c \end{pmatrix}.$$

- the arbitrariness of the labelling of the members of that set:

$$\begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix} \sim \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix}.$$

I am mainly interested in the the arbitrariness of the labelling:

$$\begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix} \sim \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix}.$$

It looks like what the first permutation does with b is what the other does with c, and vice versa. It is as if both permutations were doing exactly the same thing, except that the labels of b and c have been swapped. And well, after all the object called 'b' could have been called 'c', and conversely.

In reality, it makes absolutely no sense to mark off the objects from their names. We do not have access to a otherwise than by its name. The very distinction between objects and names is confused.

Yet he representation of a permutation involves arbitrary choices which make it appear as an invariant under typographical permutations.

So, to understand how to represent a permutation, one ought to have some prior understanding of what a permutation is.

There is no vicious circle here, still things turn out to be trickier than one could expect them to be.

The set (group) of all permutations on a set X with n elements is written $\mathfrak{S}_n$.
Strictly speaking, $\mathfrak{S}_n$ is defined as the group of all permutations on the particular set $\{1, 2, \ldots, n\}$.
The identification is natural, since the group of all permutations $\mathfrak{S}_X$ on *any* n-element set X is isomorphic to $\mathfrak{S}_n$.

The isomorphism, however, is not canonical. And selecting one precisely amounts to selecting a certain labelling (or numbering) of X.

In the case $X = \{1, 2, 3, \ldots, n\}$, the numbering is trivial, i.e., amounts to taking each number as its own numeral:
$1 = 1, 2 = 2, \ldots, n = n$.

So, in the usual representation of a permutation, the labels are the numerals directly corresponding to the numbers. The numbers act as their own numerals.

But, then, what happens if the numbers are confused with numerals?

It is not ununderstandable to understand numerals 1, 2, 3, . . . as indices of their respective canonical ranks, rather than the underlying individual objects that these numerals are supposed to stand for throughout the permutation.

Let Π the following permutation:

$$\left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{array}\right).$$

Here 1, 2, 3 et 4 are not ranks, but individual objects, whose orbits are considered as Π is iterated:

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 2 & 1 \\ 1 & 2 & 3 & 4 \end{array}.$$

Now let's imagine the following heterodox interpretation, let's call it Dummy's interpretation.
Dummy understands $1, 2, 3, 4$ as ranks rather than as objects, and accordingly carries out the iteration of Π in this way:

$$R_1 = 1 \;\; 2 \;\; 3 \;\; 4 = 1.1 \;\; 2.1 \;\; 3.1 \;\; 4.1$$

$$R_2 = 4 \;\; 3 \;\; 1 \;\; 2 = 1.2 \;\; 2.2 \;\; 3.2 \;\; 4.2 \; .$$

The second row R_2 is understood as reindexing the ranks: 4 becomes the "new" 1, so to speak, that is, the "new" number 1 position. Similarly 3 becomes the "new" 2, and so on. In this perspective, 1 is not only 1, but first and foremost the index of the first position in the current row, and the permutation is viewed as a perturbation of the ranks in reference to $\{1, 2, \ldots, n\}$.

$$\Pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}$$

According to Dummy's reading, the calculation of the third row R_3 goes that way:

> "4 *becomes* 2*, but since* 2 *now is* 3 *(because* $2.2 = 3$*),* 4 *finally lands on* 3*. So* 3 *will be the first numeral on* R_3*."*

The basic principle of that heterodox interpretation is that the 4 on the first row R_1 becomes 2 on the second row R_2, but 2 as understood precisely according to the new code set by R_2, namely as $2.2 = 3$. In the same way, 3 becomes 1 with $1.2 = 4$.
One finally gets:

$$R_3 \quad = \quad 3 \;\; 4 \;\; 2 \;\; 1 \quad = \quad 1.3 \;\; 2.3 \;\; 3.3 \;\; 4.3 \quad ,$$

whereas the correct iteration of Π of course is:

$$\begin{array}{lllll} R_1 & 1 & 2 & 3 & 4 \\ R_2 & 4 & 3 & 1 & 2 \\ R_3 & 2 & 1 & 4 & 3 \end{array} \quad .$$

In other words, Dummy takes the numbers on which the permutation acts to be floating ranks, whose counterpart is reset at each step.

Let's give another example of Dummy's misinterpretation. The correct iteration of the cycle $(12345) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$ is:

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{array} .$$

The first three rows, in Dummy's version, are on the contrary:

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 4 & 5 & 1 & 2 & 3 \end{array} .$$

Indeed, the first item in the third row is not 3, but 3 as a rank, i.e., the holder of the third position as determined in the second row, namely 4.

The full iteration of the cycle (12345), in Dummy's interpretation, is:

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 4 & 5 & 1 & 2 & 3 \\ 4 & 5 & 1 & 2 & 3 \\ 2 & 3 & 4 & 5 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{array} .$$

Obviously, Dummy's heterodox interpretation is wrong, based on a conflation of numbers as objects (as when one says that 4 "becomes" 2) and numbers as contextual indices of positions in a given row (as when one says that 3 has become the new 2 in the context of R_2).

Dummy's interpretation is wrong, but not nonsensical.
This is a case of misunderstanding about mathematics of which one can find some kind of mathematical explanation.

Mathematical understanding relies on a correct understanding of how mathematical labelling works.

Labellings, numberings, parameterizations in general turn out to be pervasive throughout mathematics, as devices to "set ideas."
I will use the general term settings to refer to them.

Settings are actually so pervasive and so important to grasp that any correct account of mathematical objectivity has to mention them.

Emergency plan

- Shapiro and Keränen do not recognize settings as the fundamental intermediate level between structures and systems, hence their endless controversy.
- Settings fall to the category of *presentations*, both in the mathematical and in the philosophical sense.
- Hypothesis: presentations (settings) in the mathematical sense (in the sense of the presentation of a group, in particular) and presentations in the intensional or phenomenological sense (Frege's *Art des Gegebenseins*, Brentano's *Modus des Vorstellens*, Husserl's *Vorstellungen*) are related and call for a unified treatment.

There are numerous examples of settings in mathematics. Among them, the choice of an origin and coordinate axes in geometry (shift from a vector space to an affine space).

Each setting involves arbitrary choices, and so in this sense is "variable" to the extent that it could have been different. But this variability is only virtual: Once set, a setting cannot change. Otherwise confusions follow.

A setting is what endows a structure with a certain angle, so that the structure becomes rigid (i.e., deprived of any nontrivial automorphism). A setting kills all parasitic symmetries.

Back to the Identity Problem (Keränen *vs* Shapiro controversy)

A setting is neither the structure itself, just as it is, nor a mere particular system instantiating the structure.

A setting is not a structure: The labelling of the members of a given system is not relevant to study the formal relations that hold between them.

Still, a setting is not a mere system, because two different settings of the same structure do not differ as systems.

What does the controversy consist in?

I think that it is this:

- Shapiro thinks that a setting is already a structure, so that the identity problem cannot even be raised.

  Shapiro thinks that presenting a structure through a setting of it is only getting to the structure itself.
  So a structure contains in itself the differentiation of its objects (places): We can see that distinct items are distinct, "We know what identity is."

- Keränen thinks that a setting is still a system, so that shifting to the structure requires to get rid of any means to distinguish numerically distinct items.

  In Keränen's reckoning, on the contrary, a setting cannot give but itself, so that the need of a setting to reach a structure simply proves that the intended structure does not exist.

In Shapiro's view, a setting "enriches" a structure, but as being part of it, and does not add anything.

In Keränen's view, a structure endowed with a setting cannot be but a system, an instance of the structure.

The mistake shared by both Shapiro and Keränen is to fail to distinguish between structures, systems *and settings*.

A nontrivial automorphism of a structure is always an isomorphism between two settings associated to the structure.

This solves the identity problem, but at the cost of acknowledging that a structure is never accessible "as such."

Mathematical structures can never be given "in themselves," but only through *modes of presentation*, of which settings are a fundamental component.

Settings are instrumental to setting ideas. They are parameterizing devices without whose bias the intended structure cannot be reached, i.e. cognitively handled.
This is shown by our example of the symmetric group $\mathfrak{S}_3$.

Bringing up the notion of mathematical "setting" or of mathematical "presentation" is a general way to single out the pervasive use of such devices throughout mathematics.

Settings and presentations do not boil down to sub-mathematical conditions of concrete mathematical activity, such as the actual drawing compared to the geometrical theorem.
They are truly mathematical in nature.
They are integral to mathematical understanding (and to mathematical objectivity).

The concept of presentation

It has both a mathematical side and a philosophical side.

Mathematical side:

- all systems of parameterization already discussed
- Presentation of a group:
  $[a, a^n = 1]$ is (a presentation of) the cyclic group $\mathbb{Z}/n\mathbb{Z}$.
- projective resolution of a module

Philosophical side:

- Brentano: "Every intentional experience is either a presentation (*Vorstellung*) or is founded upon a presentation."
- Frege: The very phrase "mode of presentation" [*Art des Gegebenseins*] is mentioned by Frege in the context of the distinction that he drew between sense and denotation, in order to account for the nontrivial nature of mathematical identities.
- Husserl, *Fifth Logical Investigation*: Husserl objects to the idea that, underlying every consciousness, there are acts of a special kind which merely present objects as the content for other, higher-order acts of consciousness.

Frege, "Über Sinn and Bedeutung:"

If the sign 'a' is distinguished from the sign 'b' only as object (here, by means of its shape), not as sign (i.e. not by the manner in which it designates something), the cognitive value of $a = a$ becomes essentially equal to that of $a = b$, provided $a = b$ is true. A difference can arise only if the difference between the signs corresponds to a difference in the mode of presentation of that which is designated. Let a, b, c be the lines connecting the vertices of a triangle with the midpoints of the opposite sides. The point of intersection of a and b is then the same as the point of intersection of b and c. So we have different designations for the same point, and these names ('point of intersection of a and b', 'point of intersection of b and c') likewise indicate the mode of presentation [Art des Gegebenseins]; and hence the statement contains actual knowledge.

The choice that Frege made of the term "presentation" could certainly be driven back to two opposite sources:

- the work of Franz Brentano in psychology, in particular his book *Psychology from an Empirical Standpoint* (1874), where the notion of "mode of presentation" (*Modus des Vorstellens*) is ubiquitous
- the foundation of the theory of group presentations by Walther von Dyck (a student of Felix Klein) in his article "Gruppentheoretische Studien," published in 1882 in the *Mathematische Annalen*.

Conclusion

1. The "identity problem" stems from the confusion of mathematical settings with either structures or systems. Shapiro lapses in the first kind of confusion, Keränen in the second kind.
2. Mathematical settings (labellings, parameterizations, and so on) are pervasive and integral to mathematical objects.
3. Mathematical understanding begins with a correct understanding of the behavior of mathematical settings.
4. Mathematical understanding should be understood so as to make it possible to understand mathematical misunderstanding. (Dummy is not a freak.)
5. Mathematical understandingand mathematical objectivity should not be philosophically analyzed separately.

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