Homotopy Model Theory

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Homotopy theory dramatically entered the scene of logic through the connections that have been made between Martin-Löf type theory and model categories.
This is Homotopy Type Theory.

I would like to show that logic can be connected with homotopy theory through model theory.
That would be Homotopy Model Theory.

Starting point

The notion of boundary can easily be transposed in the context of first-order logic, formulae being conceived of as chains (in the sense of a formal sum of faces and, in homology, in the sense of a chain complex).

The boundary of a given formula $\phi(v_0, v_1, \ldots, v_n)$ with exactly $v_0, v_1, \ldots, v_n$ as free variables, can be defined as follows:

$$\partial\phi := \bigwedge_{i=0}^{n-1} \neg^i \forall x \phi(v_0, \ldots, v_{i-1}, x, v_{i+1}, \ldots, v_{n-1}).$$

More specifically

Let's consider a fixed first-order language L with equality

- which contains at least a unary quantifier Q
- whose free variable symbols are exactly 'v_i', $i \geq 0$
- and whose bound variable symbols are exactly 'x', 'y', 'z', and so on.

Formulae of L will be taken up to bound variable renaming, but not up to free variable renaming.

Formulae will be so written that any variable with n free variables has exactly $v_0, v_1, \ldots, v_{n-1}$ as free variables.
Otherwise put, an L-formula (in the usual sense) such as $(P(v_0) \wedge \neg P(v_2))$ will not be deemed to be a well-formed formula.

Formulas as chains

Let's get back to

$$\partial\phi := \bigwedge_{i=0}^{n-1} \neg^i \forall x \phi(v_0, \ldots, v_{i-1}, x, v_{i+1}, \ldots, v_{n-1})$$

and let's consider, for instance, a formula $\phi(v_0, v_1, v_2)$ with exactly three free variables v_0, v_1 and v_2.

One gets, successively:

- first, the conjunction of $\forall x\phi(x, v_0, v_1)$, $\neg\forall x\phi(v_0, x, v_1)$ and $\forall x\phi(v_0, v_1, x)$;
- then, the inconsistent conjunction of the following six formulae: $\forall y\forall x\phi(x, y, v_0)$ and $\neg\forall y\forall x\phi(x, v_0, y)$, $\forall y\neg\forall x\phi(y, x, v_0)$ and $\neg\forall y\neg\forall x\phi(v_0, x, y)$, $\forall y\forall x\phi(y, v_0, x)$ and $\neg\forall y\forall x\phi(v_0, y, x)$.

So in the end:

$$\partial(\partial\phi) \equiv \bot$$

Simplicial ideas

Let F_n be the set of formulae of L with exactly $v_0, \ldots, v_n$ as free variables.
(F_{-1} may be defined as the set of all sentences of L.)

The two following applications $d_i : F_n \to F_{n-1}$ and $s_i : F_n \to F_{n+1}$ can then be defined:

$$d_i(\phi(v_0, \ldots, v_n)) = Qx\phi(v_0, \ldots, v_{i-1}, x, v_i, \ldots, v_{n-1})$$

$$s_j(\phi(v_0, \ldots, v_n)) = ((v_j = v_{j+1}) \to \phi(v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_{n+1})).$$

Provided that the quantifier Q satisfies, for every formula φ

(a) $QyQx\varphi(y, x, \vec{u}) \equiv QyQx\varphi(x, y, \vec{u})$

(b) $Qx((x = y) \to \varphi(y, \vec{u})) \equiv \varphi(x, \vec{u})$,

the following equalities (up to logical equivalence) are verified:

- $d_i d_j = d_{j-1} d_i$ for $i < j$
- $s_i s_j = s_{j+1} s_i$ for $i < j$
- $d_i s_j = \begin{cases} s_{j-1} d_i & \text{for } i < j \\ \text{id} & \text{for } i = j \text{ or } i = j+1 \\ s_j d_{i-1} & \text{for } i > j+1 \end{cases}$

Definition

A simplicial set is a sequence $(X_n)_{n\geq 0}$ of sets, together with maps $d_i^n : X_n \to X_{n-1}$ $(0 \leq i \leq n)$ and $s_j^n : X_n \to X_{n+1}$ $(0 \leq j \leq n)$, for each n, satisfying the simplicial identities:

$$\begin{cases} d_i d_j = d_{j-1} d_i & \text{if } i < j \\ d_i s_j = s_{j-1} d_i & \text{if } i < j \\ d_j s_j = d_{j+1} s_j = \text{id} \\ d_i s_j = s_j d_{i-1} & \text{if } i > j+1 \\ s_i s_j = s_{j+1} s_i & \text{if } i \leq j \end{cases}$$

Proposition

$F_*^Q = \langle F_n, (d_i^n)_{0\leq i\leq n}, (s_j^n)_{0\leq j\leq n}\rangle_{n\in\mathbb{N}}$ is a simplicial set.

Simplicial topology

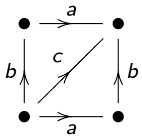
CW complexes were introduced in algebraic topology in order to analyze "nice" topological spaces as reconstructible from elementary "cells," as the result of glueing cells along other cells of smaller dimension.

A simplicial complex X is just a recipe for joining polyhedra together so as to obtain a CW complex $|X|$, called the topological realization of X.

Conversely, given a CW complex X, a simplicial complex whose realization is (up to homeomorphism) identical with X can be conceived of as a triangulation of X.

Simplicial complexes can be described diagrammatically.

For instance,



is the representation of the two-dimensional torus $\mathbb{T}^2$.

(The tags on arrows indicate how to glue edges together and thus reconstruct the torus as if through a paper folding exercise.)

Simplicial sets were introduced as a means to encode the purely combinatorial properties of the construction and shape of CW complexes.
They retain the mere skeleton of the building pattern of CW complexes and are deprived of any topology, in contrast with CW complexes.

Yet, quite surprisingly, simplicial sets are sufficient to capture most features relevant to homotopy theory.
Simplicial topology is the conceptual core of modern homotopy theory.

Owing to both their combinatorial nature and their topological meaning, their application to logic should come as no surprise, and supply an interesting connection between the combinatorial aspect of logical syntax and the use of topological methods in model theory.

In this simplicial perspective ("formulae as chains"):

- Any (generalized unary) quantifier Q satisfying conditions (a) and (b) becomes a "face operator" (i.e., the sequence (d_i^Q))
- while (s_j) appears to be the corresponding sequence of "degeneracy operators."

What about connectives?

Introducing bisimplicial objects, we are in a position to characterize the usual connectives.
Indeed, the commutative diagram:

$$\begin{array}{ccc} F_{m,p} & \xrightarrow{\;c\;} & F_n \\ {\scriptstyle \langle Q,Q' \rangle}\downarrow & & \downarrow{\scriptstyle Q''} \\ F_{m-1,p-1} & \xrightarrow[\;c'\;]{} & F_{n-1} \end{array}$$

is a way to express $Q''x(\phi c \psi) \equiv (Qv_i\phi)c'(Q'v_i\psi)$ for any formulae $\phi \in F_m$, $\psi \in F_p$ (here $n = \max(m,p)$).

We have that $\wedge$ is characterized by:

$$
\begin{array}{ccc}
F_{m,p} & \xrightarrow{\;c\;} & F_n \\
\Big\downarrow{\scriptstyle\langle\forall,\forall\rangle} & & \Big\downarrow{\scriptstyle\forall} \\
F_{m-1,p-1} & \xrightarrow[c]{} & F_{n-1}
\end{array}
$$

and $\vee$ by:

$$
\begin{array}{ccc}
F_{m,p} & \xrightarrow{\;c'\;} & F_n \\
\Big\downarrow{\scriptstyle\langle\exists,\exists\rangle} & & \Big\downarrow{\scriptstyle\exists} \\
F_{m-1,p-1} & \xrightarrow[c']{} & F_{n-1}
\end{array}
$$

Negation becomes a simplicial morphism between $F_*^{\forall}$ and $F_*^{\exists}$. Indeed,

$$\begin{array}{ccc} F_n & \xrightarrow{\neg} & F_n \\ \downarrow{\scriptstyle\forall} & & \downarrow{\scriptstyle\exists} \\ F_{n-1} & \xrightarrow[\neg]{} & F_{n-1} \end{array}$$

commutes, and in fact negation is characterized by that condition.

Semantics

Definition

Let T be a fixed L-theory. Given a model M of T,

$$M_* := F_*^{\exists,M} = \langle D_n(M), (\exists_i^{n,M})_{0\leq i\leq n}, (s_j^{n,M})_{0\leq j\leq n}\rangle_{n\in\mathbb{N}}$$

where:

- $D_n(M)$ (for $n \geq 0$) is the set $\mathrm{Def}_{n+1}(M)$ of all definable subsets of $|M|^{n+1}$ and $D_{-1}(M)$ is the theory $Th(M)$ of M
- $\exists_i^{n,M} : D_n(M) \to D_{n-1}(M)),\ A = \{\vec{a} \in |M|^{n+1} : M \vDash \phi_A(v_0, \ldots, v_n)[\vec{a}]\} \mapsto \{\vec{a'} \in |M|^n : M \vDash \exists x\ \phi_A(v_0, \ldots, v_{i-1}, x, v_i, \ldots, v_{n-1})[\vec{a'}]\}$ are the face operators
- $s_j^{n,M} : D_n(M) \to D_{n+1}(M),\ A \mapsto \{(\vec{x}, y) : \vec{x} \in A \text{ and } y = x_j\}$ are the degeneracy operators.

Proposition

For any M, M_* is a simplicial set.

General remark

In general, a model cannot be reconstructed from the hierarchy of its definable subsets. Two models with the same definable subsets (up to isomorphism) need not be elementarily equivalent.

Conversely, any two elementary equivalent L-structures M and N have isomorphic hiearchies of definable subsets: For any L-formula ϕ, it suffices to take $\phi^M \mapsto \phi^N$.
Yet this latter isomorphism does not come from any actual map between M and N.

Basic notions of model theory: substructure (embedding)

and elementary substructure (elementary embedding).

Tarski-Vaught test:
M being a substructure of N, M is an elementary substructure of N iff, for any formula $\phi(x, a_1, \ldots, a_n)$ with parameters $a_1, \ldots, a_n$ from M,
$N \models \exists x \phi(x, a_1, \ldots, a_n)$ implies that there is $a \in |M|$ such that $N \models \phi(x, a_1, \ldots, a_n)[a]$.

Definition

Given two simplicial sets X and Y, a simplicial map $f : X \to Y$ is a family of maps $f_n : X_n \to Y_n$ $(n \geq 0)$ which commute with the operators d_i and s_j, e.g.:

$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & Y_n \\ {\scriptstyle d_i^X}\downarrow & & \downarrow{\scriptstyle d_i^Y} \\ X_{n-1} & \xrightarrow[f_{n-1}]{} & Y_{n-1}. \end{array}$$

Definition

Let M be a substructure of an L-structure N. For each $n \geq 0$, the restriction map r_n sends ϕ^N to $\phi^N \cap |M|^{n+1}$, for each formula $\phi(v_0, \ldots, v_n) \in F_n$.

Theorem

A substructure M of an L-structure N is an elementary substructure of N iff the restriction maps r_n induce a well-defined simplicial map $r_* : N_* \to M_*$.

Remark: The restriction maps r_n making up a simplicial map from N_* to M_* is the direct expression that the Tarski-Vaught test is met.

$$\begin{array}{ccc} D_n(N) & \xrightarrow{r_n} & D_n(M) \\ \downarrow{\scriptstyle \exists_i^{n,N}} & & \downarrow{\scriptstyle \exists_i^{n,M}} \\ D_{n-1}(N) & \xrightarrow[r_{n-1}]{} & D_{n-1}(M) \end{array}$$

Corollary

The mapping $(-)_*$ *is a contravariant functor from the category of L-structures and elementary embeddings, to the category of simplicial sets and simplicial maps.*

Each elementary embedding $f : M \to N$ gives rise to $f_* : N_* \to M_*$ in a functorial way, with
$f_n : \phi^N \mapsto \{\vec{a} \in |M|^{n+1} : N \vDash \phi[f(\vec{a})]\} = \phi^M$.

Let M be an elementary substructure of N, and let's suppose that $|M|$ is definable in N, by a formula $\underline{M}(x)$ of L. For any $\phi \in F_n$, $\phi^{\underline{M}}$ is the formula $(\underline{M}(v_0) \wedge \ldots \wedge \underline{M}(v_n) \wedge \phi^{(\underline{M})})$, where $\phi^{(\underline{M})}$ means the relativization of ϕ to $\underline{M}$. This relativization allows one to define extensions $e_n : \phi^M \in D_n(M) \mapsto (\phi^{\underline{M}})^N \in D_n(N)$, and thus a morphism $e_* : M_* \to N_*$.

Definition

Given two simplicial sets X and Y, a *retraction* of Y over X consists of a pair $\langle f, g\rangle$ of simplicial morphisms $f : X \to Y$ and $g : Y \to X$, such that $g \circ f = \mathrm{id}_X$ (which implies that f is a monomorphism).

Theorem

Let M be an elementary substructure of N. Then the domain $|M|$ of M is definable in N iff $\langle e_*, r_*\rangle$ defines a retraction of N_* over M_*.

Types

Definition

Given a complete theory T in L (for instance, the theory $T(M)$ of some L-structure M), an n-type is a set of formulae with exactly $v_0, \ldots, v_n$ as free variables, which is consistent with T.
The set of all n-types is written $S_n(T)$.

Proposition

$S_*(T) = \langle S_n(T), (\exists_i^n)_{0 \leq i \leq n}, (s_j^n)_{0 \leq j \leq n} \rangle_{n \in \mathbb{N}}$
('$\exists_i$' being just a shorthand for $\exists v_i$)
is a simplicial set.

Standing back

Common place (common practice): The philosopher working in philosophical logic is not supposed to delve into too hard technicalities.

However, shying away from the mathematical tools resorted to by the philosopher detracts from philosophical rigor.

It also departs from the very tradition of logic (from Frege onward, at least):

- Frege's *Begriffsschrift* = formal language "modeled upon that of arithmetic."
- Russell's logic = formal language modeled upon that of the theory of relations.
- Tarskian logical semantics = formal language modeled upon that of set theory.

Mathematical "tools"

The philosopher has to get his hands dirty.

The philosophical logician catches up there with the philosopher of mathematics.
(Algebra and topology cannot be missed as sources and tools for logic, and for model theory in particular. Still, algebraic topology has not yet been centrally and explicitly implemented at the elementary level of basic logical notions.)

Algebraic topology provides the philosopher with the means to reconsider quantifiers, by conceiving of them as "face maps."

Another quick example: fibered categories

A fibered category generalizes the notion of surjective map.

Definition

A fibered category is a functor $p : F \to C$ such that, for any object $x \in F$ and any arrow $u : X' \to p(x)$ in C, there is an arrow $\widehat{u} : x' \to x$ in F that "lifts" u.

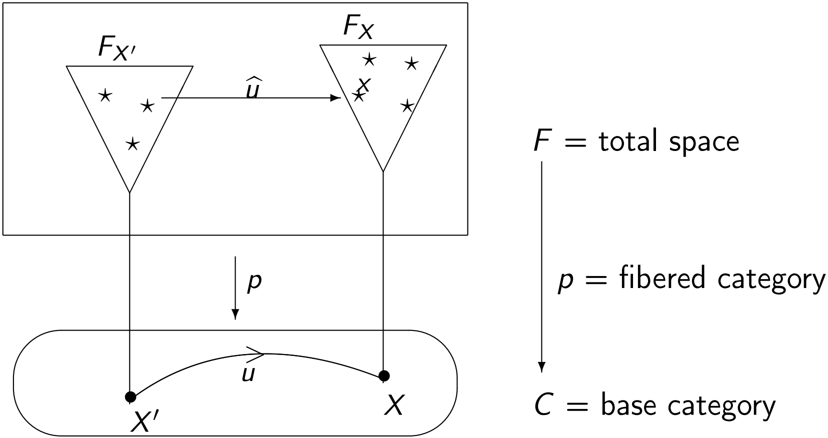
The category C is called the "base category," F the "total space."

For each object X in the base C, the category $F_X := p^{-1}(X)$ is called the "fiber" above X, which generalizes the set-theoretic notion of pre-image.

Fundamental extra-ingredient carried by a fibered category, in comparison with surjective maps:
The base category C is endowed with some structure (as embodied by its arrows) and, owing to the correspondence $u \mapsto \widehat{u}$, the structure of the base C is reflected in a principled way by that of the total space F.

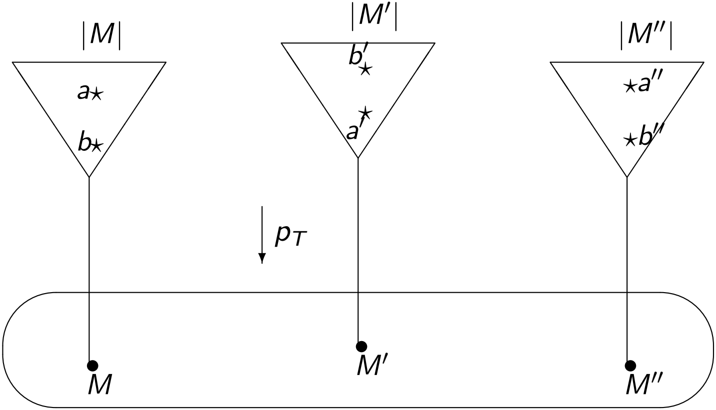
Caveat: The choice of fibered categories has nothing to do with the category theory *vs* set theory controversy.

You can think of a fibered category $p : F \to C$ "bottom up," as a mapping which assigns to each object X of C its fiber F_X.



A particular fibered category

The base category is the category $\mathcal{S}$ of all L-structures for a fixed first-order language L, and the arrows of $\mathcal{S}$ come from inverting all the L-homomorphisms between L-structures.
The fiber $\mathcal{A}_M$ above any structure M in $\mathcal{S}$ is the set $|M|^{\mathsf{Var}}$ of all assignations in the domain $|M|$ of M.
One gets a fibered category p_T, "Tarski's fibration".



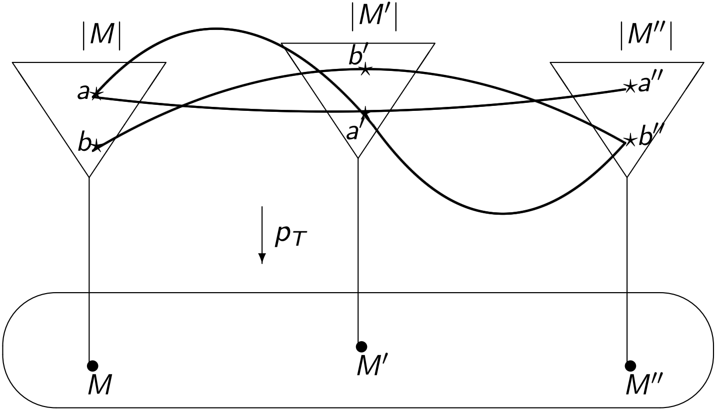
In Tarskian semantics, given a *fixed* interpretation structure $M \in S$ for a language L, each assignment gives a specific value in M to *each* variable symbol of L.

Let's turn things around: Let's consider a fixed variable symbol 'x' and consider the value that it gets in each possible structure for the language.

Definition

A generalized assignment is a map $S \times \mathrm{Var} \to \coprod_{M \in S} |M|$ that assigns to each structure $M \in S$ and each variable symbol $v \in \mathrm{Var}$ a given element of the domain of M.

If one focuses on the single variable symbol 'x', then a generalized assignment is nothing else but a choice function $\alpha \in \prod_{M \in S} |M|$, that selects a member of the domain of each structure M as the value for 'x' in M. It is a particular section of the total space.



In a way, all the different values that an x-generalized assignment α gives to 'x' can be conceived of as a sequence of counterparts of an original value.

The sequence a', a'', ..., could be viewed as what a becomes when one shift from context D to context D', then to context D'', and so on.

Tarski's semantics is characterized by the fact that all sections are allowed.

In Tarski's semantics, the base category of Tarski's fibration does not work as a real control space.

This is NOT a problem per se. BUT:

- this raises difficulties in philosophy of language (think of rigid designators, for instance).
- this shows how the "tools" matter and influence the representation of thought an language which behooves the philosopher.

Conclusion

The philosopher is partly in charge of her mathematical tools.

There are various ways of analyzing and representing generality:

- quantifiers as face maps,
- variables as sections of Tarski's fibration.