Relativizing Tarskian Variables

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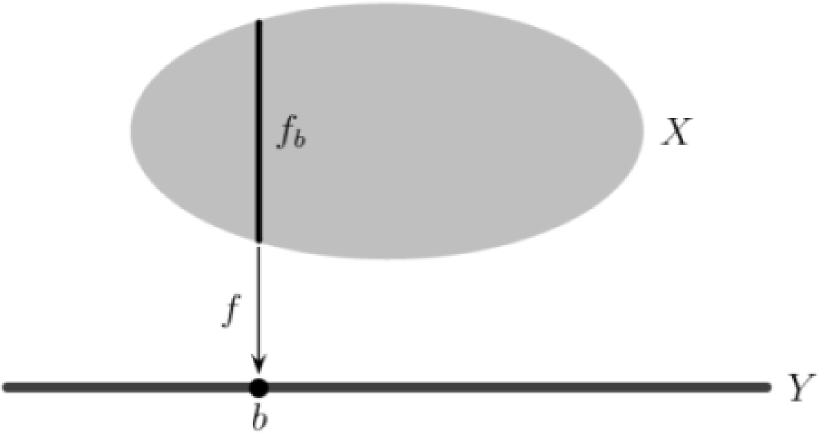
Two main goals:

- Introducing the concept of fibration (that comes from geometry) and showing that it holds out a natural way to formalize how the interpretation of variables works in Tarski's semantics.
- Relativizing Tarski's variables as "flat" variables.

Fibration = topological and categorical generalization of a surjective map.



A *category* A consists in two collections, a collection Ob A of objects $a, b, c, \ldots$ and a collection Arr A of arrows $f : a \to b$ between those objects. Examples: *Set*, *Top*.
Any category A has an *opposite category* A^o.

A *functor* $F : B \to C$ from a category B to a category C is a correspondence that sends any object b of B to some unique object $F(b)$ of C and sends any arrow $f : b \to b'$ in B to a unique arrow $F(f) : F(b) \to F(b')$ in C, so that $F(1_b) = 1_{F(b)}$ and $F(f \circ g) = F(f) \circ F(g)$.

A category is *small* if the collection of all its arrows is set-like.
Cat = category of all small categories and functors between them.

Given a category B, an *indexed category* over B is a functor $F : B^o \to Cat$. So it is a correspondence:

- sending each object b of B to a category $F(b)$, called the "fiber" above b.
- sending each arrow $u : b \to b'$ in B to a functor $F(u) : F(b') \to F(b)$, also written u^*.

So an indexed category amounts to a family that is organized in a functorial way.

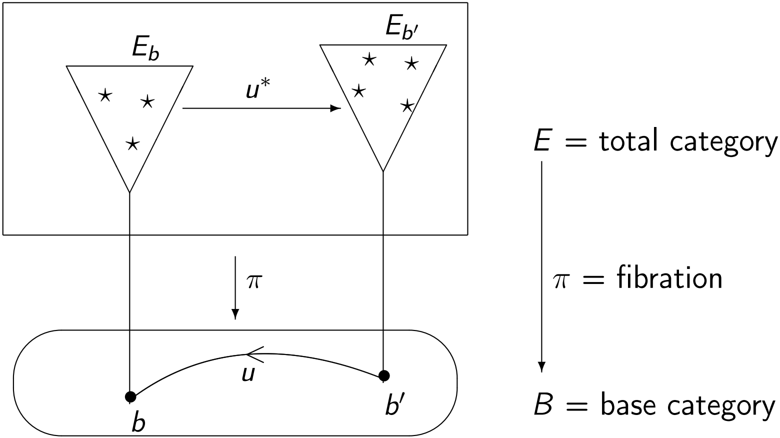
Given an indexed category F,

- B is the "base category"
- $E := \coprod_{b \in \mathrm{Ob}\ B} F(b)$ is the "total space"
- the functor $\pi_F : E \to B$ that sends any object in $F(b)$ to b, any object in $F(b')$ to b', and so on, is a *fibration*.

Conversely, any fibration $\pi : E \to B$ gives rise to the indexed category $F_\pi : b \mapsto E_b := \pi^{-1}(b)$.

- $F : B^o \to Cat \quad \rightsquigarrow \quad \pi_F : \coprod_{b \text{ in } B} F(b) \to B$
- $\pi : E \to B \quad \rightsquigarrow \quad F_\pi = (\pi^{-1}(b))_{b \text{ in } B}$

I will henceworth introduce indexed categories but present them as fibrations.



A fibration in general

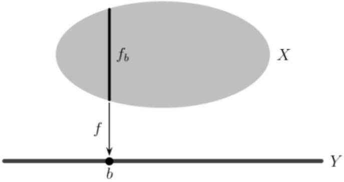
As a functor, an indexed category F sends any arrow in the base to a functor $F(u) = u^*$, called a "reindexing" functor between the corresponding fibers.

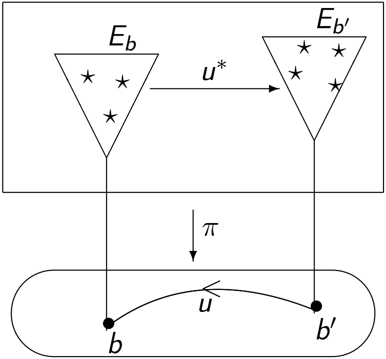
So the existence of a fibration requires then some *systematic connection* between the relations between any two points in the base (arrows u), and the relations between the corresponding fibers in the total space (functors u^*).

Owing to that systematic correspondence, one can get a sense here that the base category of a fibration works as a *control space*. This is how fibrations generalize surjections.

The base category can be endowed with any kind of structure and that the connections between the fibers must "lift" that structure and mirror it back to the total space.

Surjection vs Fibration:



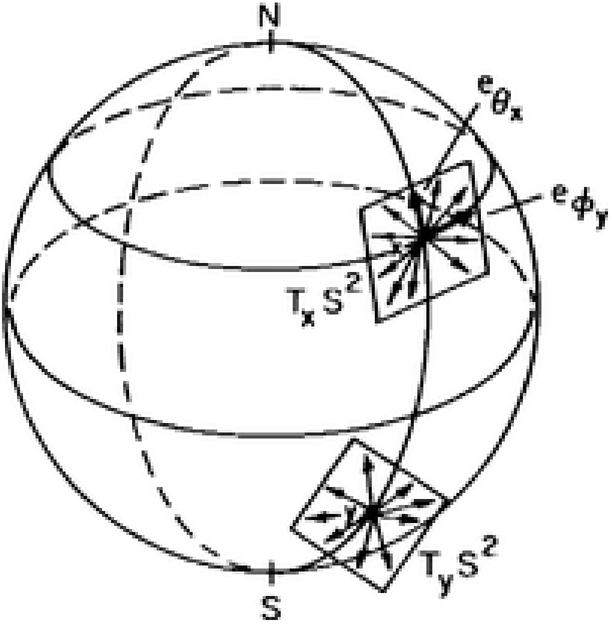


E = total category

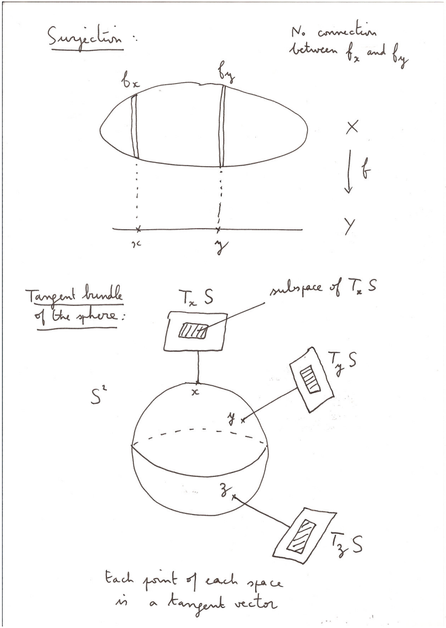
π = fibration

B = base category

Tangent planes to the sphere:

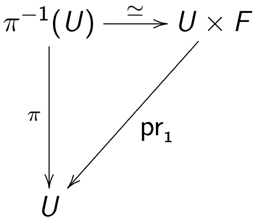


Tangent bundle and Distribution:



Special case of fibration: Fiber bundle.

The base B as well as the total space E are topological spaces. The fibration $\pi : E \to B$ is a continuous surjective map such that, for any $x \in B$, the bundle of all the corresponding fibers $\pi^{-1}(y)$ in a neighborhood U of x make up a kind of smooth cylinder above x:



F = pattern common to all the fibers.

If the same group acts upon each fiber, one gets a “principal bundle”.

To sum up, a fibration is a way to enrich a surjection with some structure: structure in the basis and structure both inside each fiber and between the fibers.

Now, my claim is:

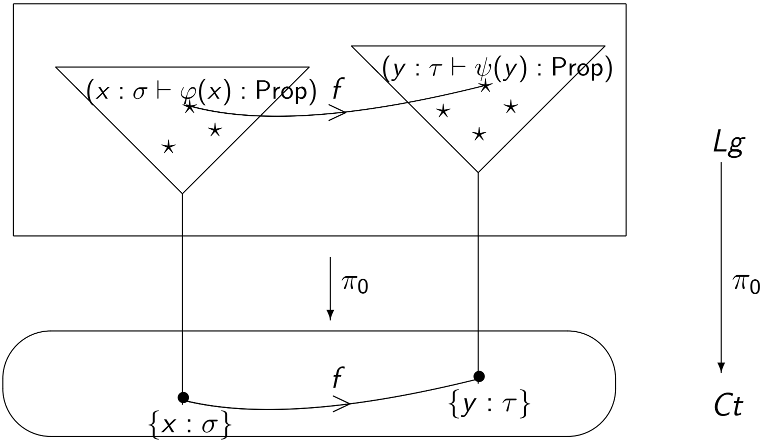
- that the concept of fibration can be applied to Tarski's semantics;
- that the framework of fibrations is the natural framework to enrich Tarski's semantical scheme for variables with additional structure.

The use of fibrations in logic is not unprecedented:

- Syntactic fibrations for type-theoretic systems
- Lawvere's fibration for quantification within a domain.

In the case of a syntactic fibration, the objects of the base category are logical contexts (declarations of variables) and the fiber above each context consists in all the logical judgments (sequents) that can be expressed in that context.

A logical calculus then is nothing else but a set of connections between judgments lying above (possibly) different contexts, i.e. a set of connections between fibers.



Conditions on f:

- $x : \sigma \vdash f(x) : \tau$
- $x : \sigma \parallel \varphi(x) \vdash \psi[f(x)/y]$

The syntactic fibration π_0

In a syntactic fibration, structure comes from the logical rules. The admissible moves along the base category and, accordingly, along the total space, are constrained.
But this is syntactic stuff, and has nothing to do with Tarski's semantics.

Lawvere's fibration has to do with Tarskian semantics. But it is implicitly confined to a given domain.

My goal: describing Tarski's logical semantics as a fibration whose base is made up by *all* the interpretations of a given first-order language.

The typical clause for the interpretation of quantification in Tarski's semantics is, for any first-order structure M and any assignment σ of values to the variables of the language:

$M \vDash \forall x \phi(x)[\sigma]$ iff, *for any assignment* σ' that differ with σ at most at x, $M \vDash \phi(x)[\sigma']$.

Variables of the language are replaced with variables of assignments.

Tarski's fibration

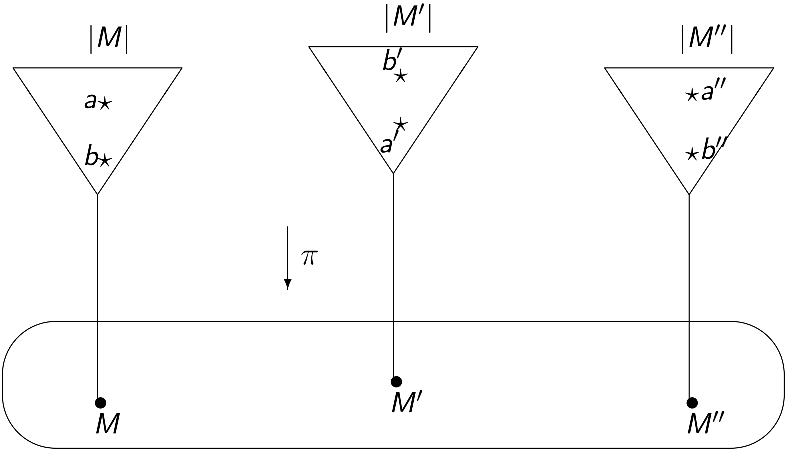
Base category $\mathcal{S}$:

- objects are all structures for a fixed first-order language L;
- there is an arrow $\bar{f} : N \to M$ in $\mathcal{S}$ each time there is an L-homomorphism from M to N.

- $F(M) = |M|^{\mathrm{Var}}$ (set of all assignations in M)
- $F(\bar{f}) = \bar{f}^* : \sigma \in |M|^{\mathrm{Var}} \mapsto f \circ \sigma \in |M_1|^{\mathrm{Var}}$.

This defines an indexed category $F : \mathcal{S}^o \to Cat$.

Hence a fibration $\pi : \mathcal{A} \to \mathcal{S}$, where $\mathcal{A}$ is the union of all $|M|^{\mathrm{Var}}$, $M \in S$.
π = **Tarski's fibration**.



Tarski's fibration π

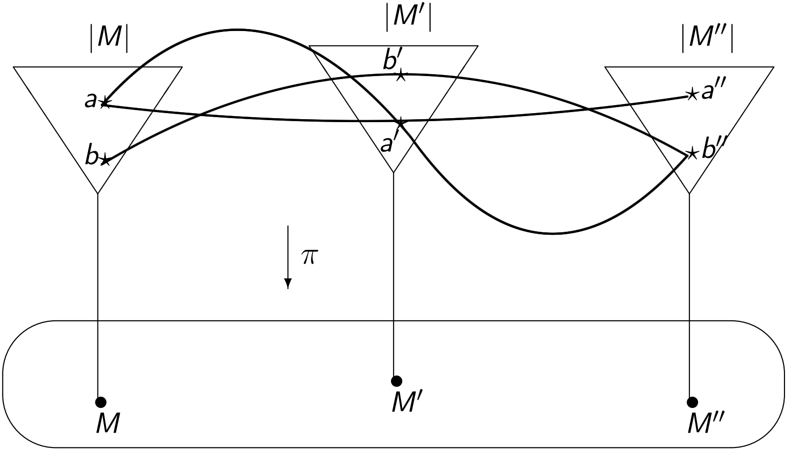
Generalized assignments

In Tarskian semantics, given a *fixed* interpretation structure $M \in S$, each assignment gives a specific value in M to *each* variable symbol of L.

Let's turn things around: let's consider a fixed variable symbol 'x' and consider the value that it gets in *each* possible structure for the language.
An "x-generalized assignment" is nothing else but a choice function $\alpha \in \prod_{M \in S} |M|$.

A *generalized assignment* does the same thing, but for all variables. It is a map that assigns to each structure $M \in S$ and each variable symbol $v \in \mathrm{Var}$ a given element of the domain of M.

A generalized assignment is nothing else but a **section** of Tarski's fibration.
Section of a fibration = successive choice of a specific object in *each* fiber.
It is a particular crossing of the total space.



Tarski's fibration π: all sections (thick curves) are allowed.

No constraint is placed on sections of Tarski's fibration.
At each point, in Tarski's fibration, the choice of the value of 'x' in any interpretation structure M is completely free. This is why Tarski's semantics constitutes a kind of limit case.

Structure comes from the way in which a total range of values (a range of ranges, or "total space") is organized by prescribed generalized values (prescribed crossing sections).

The smaller the leeway in the choice of single values to introduce a section, the more constrained are the generalized values, and the more *structured* is the fibration.

Natural question: How to enrich Tarski's fibration with further structure?

Dynamic logic conceives of assignments as being possible worlds. Existential quantification over x becomes a possibility operator:

$M, \sigma \vDash \Diamond_x \varphi$ iff there exists θ such that $\sigma R_x \theta$ and $M, \theta \vDash \varphi$.

Then it is possible to consider that the assignments which are genuinely accessible from a given assignment make up a proper subset of $|M|^{\mathsf{Var}}$ only.

J. van Benthem, *Exploring Logical Dynamics* (1996):
A *generalized assignment model* is a couple $\langle M, \mathbb{V} \rangle$, where M is a regular L-structure and $\mathbb{V}$ is a selection of "available assignements" in M.
First-order evaluation then goes:

$\langle M, \mathbb{V} \rangle, \alpha \vDash \exists x \phi$ iff, for some $d \in |M|$, $\alpha^x_d \in \mathbb{V}$ s.t. $\langle M, \mathbb{V} \rangle, \alpha^x_d \vDash \phi$.

The selection $\mathbb{V}$ can be used to reflect *dependencies* between variables. For example, one can force 'x' and 'y' to have always different values.

J. van Benthem, *Exploring Logical Dynamics*, p. 177:

> *Standard models are then 'degenerate cases' where all dependencies between variables have been suppressed.*

J. van Benthem & N. Alechina, "Modal quantification over structured domain" (1997):

The Tarskian truth condition for the existential quantifier reads as follows:

$$M, [\overline{e}/\overline{y}] \vDash \exists x \varphi(x, \overline{y}) \;\; \Leftrightarrow \;\; \exists d \in D : M, [d/x, \overline{e}/\overline{y}] \vDash \varphi(x, \overline{y})$$

This may be viewed as a special case of a more general schema, when the element d is required in addition to stand in some relation R to $\overline{e}$ *– where R is a finitary relation* **structuring** *the individual domain D:*

$$M, [\overline{e}/\overline{y}] \vDash \Diamond_x \varphi(x, \overline{y}) \; iff \, \exists d \in D : R(d, \overline{e}) \& M, [d/x, \overline{e}/\overline{y}] \vDash \varphi(x, \overline{y})$$

[...] One might read $R(d, \overline{e})$ *as*

- *d can be constructed using* $\overline{e}$*,*
- *d is not "too far" from the e's,*
- *after you have picked up e's from the domain without replacing them, d is still available,*

et cetera.

van Benthem & Alechina, continued:

> *Ordinary predicate logic then becomes the special case of* **flat** *individual domains admitting of "random access", whose R is the universal relation.*

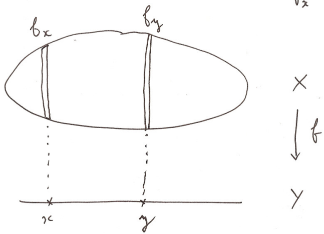
Significantly, van Benthem and Alechina spontaneously use geometric metaphors ("structuring", "flat"). But they consider only one structure M at a time.

It makes more (philosophical and mathematical) sense to consider the whole base category $\mathcal{S}$.

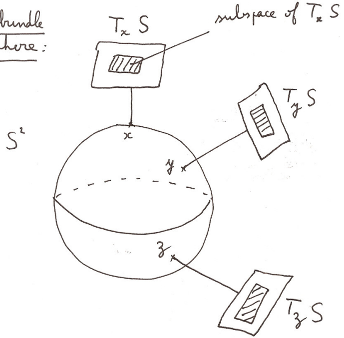
In fact, a selection of "available assignements" for *each* object in $\mathcal{S}$ coincides exactly with a *distribution* on $\mathcal{S}$.

Surjection :

No connection between f_x and f_y

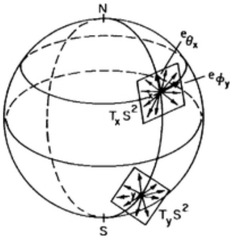


Tangent bundle of the sphere:

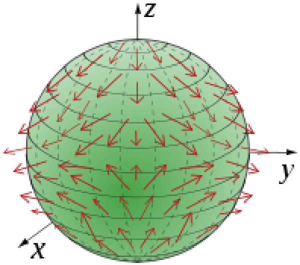


Each point of each space is a tangent vector

Tangent planes to the sphere:



Vector field on the sphere:



If one choose one vector in each tangent space to the sphere, one gets a *distribution* which is called a *vector field*.

Fixing a vector field puts a maximal constraint upon all the admissible sections.

It also puts a constraint upon all the admissible curves on the base category (*integral lines* of the vector field).

Hence a distribution is a way to build structural constraints into the movements on the manifold at stake. It is the convenient way of stating a law in modern mechanics.

Dynamic logic can be looked at as a way of structuring Tarski's fibration through its fibers.

Structure can also be given to the base category $\mathcal{S}$: "horizontal structure".
Let $\mathcal{S}_T$ be the category:

- whose objects are of the models of some first-order theory T,
- whose arrows are all *elementary extensions* reversed:
  $\bar{f} : M_1 \to M$ iff $M \xrightarrow[f]{\prec} M_1$.

Besides, to simplify matters, let's say that any formula to be considered has less than n_0 free variables (with n_0 big enough). So each fiber $|M|^{\text{Var}}$ now becomes identical with $|M|^{n_0}$.

Resulting fibration: π_T (distinct from Tarski's fibration π).

Now, any formula $\varphi(x_1, \ldots x_{n_0})$ of L induces a distribution: for each M, the subset of $|M|^{n_0}$ made up of all the tuples that satisfy φ in M.
So a formula φ becomes the analog of a *differential form*:
$\varphi_M : \vec{a} \mapsto 1$ if $M \vDash \varphi\ [\vec{a}]$, else 0.

For any (partial) section α of π_T, the *generalized satisfaction* of a formula φ by α is readily definable:

$\alpha \vDash_g \varphi(\vec{x})$ iff, for any $M \in \mathrm{dom}(\alpha)$, $M \vDash \varphi(\vec{x})$ $[\alpha(M)]$.

Let's restrict sections of π_T to partial sections along *elementary curves*, i.e. elementary chains $M_0 \prec M_1 \prec \ldots M_k \prec \ldots$

Let's say then that a section α is *elementary* if it is a section along an elementary curve $M_0 \xrightarrow[f_0]{\prec} M_1 \xrightarrow[f_1]{\prec} \ldots \xrightarrow[f_{k-1}]{\prec} M_k$, such that $\alpha(M_{i+1}) = \bar{f}_i^*(\alpha(M_i))$ for each $i \geq 0$.

For any elementary section α of π, one has that:
for any formula φ, either $\alpha \vDash_g \varphi$ or $\alpha \vDash_g \neg\varphi$.

Let $\sim$ be the "indiscernibility relation" between n_0-tuples:

$\vec{a} \sim \vec{a'}$ iff $\vec{a}$ and $\vec{a'}$ belong to the same model M and satisfy exactly the same formulas in M. Then:

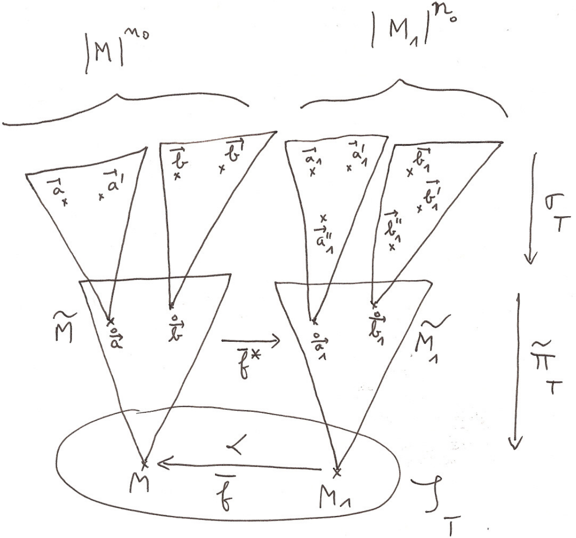
- for any model M of T, let $G(M)$ be $\tilde{M} := |M|^{n_0}/\sim$
- for any $\bar{f}$ in $\mathcal{S}_T$ (corresponding to some elementary embedding $f : M \to M_1$) let $G(f) = \bar{f}^*$ be the functor that sends any indiscernibility class $\mathring{\vec{a}}$ in $\tilde{M}$ to $f(\mathring{\vec{a}})$ in $\tilde{M}_1$.

The functor G is an indexed category.
So, writing $\tilde{A}$ for the disjoint union of all $\tilde{M}$'s, one has a fibration $\tilde{\pi}_T : \tilde{A} \to \mathcal{S}_T$.
In fact, $\tilde{\pi}_T$ is nothing but the fibration that one gets from π_T by passing to the quotient:

$$\pi_T = \tilde{\pi}_T \circ \sigma_T$$

where $\sigma_T : \mathcal{A} \to \tilde{\mathcal{A}}$ is the canonical quotient fibration.

$$\pi_T = \tilde{\pi}_T \circ \sigma_T$$



Remark: A pure n_0-type in $S_{n_0}(T)$ is nothing but a partial section of $\tilde{\pi}$.

Given a language L and an L-structure M, any maximally consistent set of formulas in L(M) with exactly one free variable x is called a *type*.
A type is *isolated* iff it consists in all the formulas in x that can be derived in T from a single formula $\varphi(x)$.

A type over M can be thought of as the description of an *ideal element* relatively to M.

In some cases, that ideal element is already realized in M, in some other cases it is not. If a model realizing all its ideal elements is said to be *saturated*.

The definition above carries over to the case of formulas with n_0 variables. So one gets n_0-types. The set of all n_0-types over M is written $S_{n_0}(M)$.

Given p in $S_{n_0}(M)$, a curve c in $\mathcal{S}_T$ starting from M and a tuple $\vec{a}$ in $|M|^{n_0}$ realizing p in M, there is a unique elementary section $\tilde{c}$ of π_T lifting c.
The section $\tilde{c}$ can be seen as a curve in $\mathcal{A}$ and can be referred to as the "p-lift of c passing through $\vec{a}$."

Proposition

A given model M is saturated iff, for any type p, any elementary curve starting from M can be p-lifted.

Possible results about isolated types.

Let M be a fixed saturated model. For any n_0-type p, $p^M = \{\vec{a} \in |M|^{n_0} : M \models p[\vec{a}]\}$.

The disjoint union $\coprod_{p \in S_{n_0}(M)} p^M$ is $|M|^{n_0}$ viewed as a total space over $S_{n_0}(M)$.
So we have a fibration $\tau_M : |M|^{n_0} \to S_{n_0}(M)$, that assigns each $\vec{a} \in p^M$ to p.

Besides, $S_{n_0}(M)$ constitutes a topological space, with $U_\varphi := \{p : \varphi \in p\}$ as basic opens (for all formulas φ with exactly $x_1, \ldots x_{n_0}$ as free variables).

Moreover, $|M|^{n_0}$ can be endowed with the topology generated by all the definable subsets $\varphi^M(x_1, \ldots, x_{n_0})$ of $|M|^{n_0}$.

Theorem

Let M be countable and saturated. Then, for any two tuples $\vec{a}$ and $\vec{b}$ in M that have the same type over M, there is an automorphism f of M such that $f(\vec{a}) = \vec{b}$.

Corollary

The group $Aut(M)$ of all automorphisms of M acts freely and transitively on each fiber p^M.

Proposition

A countable model M is saturated iff τ_M is a G-principal bundle (with $G = Aut(M)$).

For $M \prec M_1$, a type $p_1 \in S_{n_0}(M_1)$ is an *heir* of another type $p \in S_{n_0}(M)$ over M_1 iff

(i) the restriction of p_1 to $\mathrm{L}(M)$ is p and
(ii) for each formula $\varphi(\vec{x}, \vec{a}, \vec{b}) \in p_1$, with $\vec{a} \in M$ and $\vec{b} \in M_1$, $\varphi(\vec{x}, \vec{a}, \vec{a'}) \in p$ for some $\vec{a'} \in M$.

Any situation, with parameters in M, exhibited by p_1 over M_1 already has an instance proffered by p over M.

A theory T is *stable* iff any type in any model M of T has exactly one heir over any elementary extension of M.

Let T be a stable theory and let $\mathcal{S}_T$ be as above.

- $H(M) = \tilde{M}$
- for each $\bar{f} : M_1 \to M$ in $\mathcal{S}_T$, $H(\bar{f}) = \bar{f}^* : \tilde{M} \to \tilde{M}_1$ sends each realized type p in $S_{n_0}(M)$ to its unique heir along f in $S_{n_0}(M_1)$.

The functor H is an indexed category, whose base category is $\mathcal{S}_T$. Hence a new fibration $\lambda_T : \coprod_{M \in \mathrm{Ob}\ \mathcal{S}_T} S_{n_0}(M) \to \mathcal{S}_T$.

Proposition

T is stable iff $M \mapsto \tilde{M}$ induces a fibration (namely, λ_T).

LOGIC	GEOMETRY
Mod T with homomorphisms as arrows	Tarski's fibration π
generalized assignments (Benthem & al.)	distribution over π
Mod T with elementary embeddings as arrows	π_T and quotient fibration $\tilde{\pi}_T$
elementary chains	privileged paths in Mod T
types in $S_{n_0}(M)$	partial sections of $\tilde{\pi}_T$
M saturated	fibration τ_M
M countable and saturated	fiber bundle τ_M
T stable	fibration λ_T

Summary of the correspondences