Geometric Modal Logic

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Topic of this talk: modal iteration

Modal iteration is the superposition of modal clauses, as when some proposition is said to be necessarily necessarily true, or necessarily possibly necessarily true.

Basic idea:

ϕ could have been possible
$= \phi$ is not possible but, were the whole actual constitution of the possible replaced by another one, ϕ would become possible.

In terms of possible worlds semantics: The datum which is the collection of all possible worlds should be itself counterfactualizable
= The actual collection of all possible worlds is only one among other possible collections of possible worlds.
(See Stalnaker, "Merely Possible Possible Worlds.")

That ultimately requires to formalize the notion of higher-order possible world.
(See Kit Fine's formalization of higher-order vagueness.)

Claims:

- Modal iteration has a very strong meaning: It prompts a semantic "change of scale" (see below).
- Kripke semantics for propositional modal logic does not match that meaning.

Goals:

- challenging (and generalizing) the interpretation of iterated modalities given in the standard Kripke semantics
- putting forward a new possible worlds semantics for propositional modal logic, drawn from modern differential geometry, and establishing some completeness results within it.

Leibnizian conception of modalities:

- necessity, possibility and contingency are absolute propositional features
- possible worlds make up a closed totality
- iterated modalities are meaningless.

Leibniz's conception is tied up with the special context of his theodicy.
But actually it goes far beyond that context. E.g., *Tractatus*, 5.525:

> *Certainty, possibility or impossibility of a state of affairs are not expressed by a proposition but by the fact that an expression is a tautology, a significant proposition or a contradiction.*

(There is but one single logical space.)
Think also of Carnap's *Meaning and necessity* (modal iteration is lacking).

Predicament of modern modal semantics:

Contrary to Leibniz, it allows for modal iteration.
But, on the other hand, it retains the crucial metaphysical feature of Leibniz's conception, namely the existence of a definite closed totality of all possible worlds.

Modal iteration can be refused for very legitimate reasons.
But as soon as it is deemed to be meaningful, its meaning should be fully accounted for, which requires abandoning any fixed totality of possible worlds.

There is no reason to stick to the postulate of the totalizability of possible worlds.

A correct understanding of modal iteration even requires abandoning such a postulate, because it urges to consider different levels of possibility.

Indeed, saying that a proposition is necessarily necessarily true amounts intuitively to saying that the proposition is necessarily true whatever the range of the possible may be.

In the same way, when one says about a state of affairs that it is not, but could have been possible, one implicitly shifts from a given system of possibility into a context in which that system is relocated as being only one among others, and one claims that with respect to some other system, the state of affairs becomes possible.

In terms of possible worlds:

- truth = satisfaction in the actual world w_0.
- necessary truth = satisfaction, not only in w_0, but in any possible world in W (for simplicity's sake, every world is supposed here to be accessible from the actual one).
  Shift from w_0 to W = change of scale.
- necessarily necessary truth = ?
  Answer: It should correspond to an analogous change of scale, pointing to a set W^2 of sets of worlds, which would be to W as W is to w_0.
- And so on, in case of any further modal iteration.

Let's take stock. Alethic modalities pertain to truths as being not only factual.
Still, the range of what is possible (the collection of all possible worlds) is referred to as a kind of super-fact.

Modal logic should go beyond it: Whatever the actual range of the possible turns out to be, it should be understood as a range among other possible ones. Hence, a second-order range of possible ranges for the possible, and so on.
Taken seriously, each new application of a modal operator should be interpreted by such a change of scale.

Wanted: An open-ended collection of nested systems of possibility, where each system corresponds to a set of possible worlds lying at some level.

Syntax of propositional modal logic:

- any propositional variable is a formula
- if ϕ and ψ are formulas, then $\neg\phi$, $(\phi \wedge \psi)$ and $(\phi \vee \psi)$ also
- if ϕ, then $\Diamond\phi$ and $\Box\phi$ also.

($\Diamond$ and $\Box$ are interdefinable as dual operators.)

Modal degree of a formula: obvious.

A system of modal logic is normal if it includes the following as fundamental axioms:

- all the tautologies of nonmodal propositional logic
- axiom **K**: $\Box(p \to q) \to (\Box p \to \Box q)$
- axiom **T**: $\Box p \to p$

Inference rules: modus ponens, the necessitation rule (if ϕ is a theorem, so is $\Box\phi$) and the rule of uniform substitution (if a formula $\chi(p)$ containing p is a theorem, so is $\chi[\phi/p]$, for any formula ϕ).

The ensuing system is called T.
The addition of the axiom **4**, $\Box p \to \Box\Box p$, defines the system S4 = T + **4**.

Kripke semantics (1963)

A Kripke frame is a triple $\mathcal{F} = \langle W, R, w_0 \rangle$ (R = binary "accessibility" relation defined on W).
A Kripke model is a quadruple $\mathcal{M} = \langle W, R, w_0, V \rangle$ coming from a frame endowed with a valuation.
Satisfaction:

- $\mathcal{M}, w \models p$ iff $w \in V(p)$
- $\mathcal{M}, w \models \neg\phi$ iff $\mathcal{M}, w \not\models \phi$
- $\mathcal{M}, w \models \phi \wedge \psi$ iff $\mathcal{M}, w \models \phi$ and $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models \Diamond\phi$ iff there exists $w' \in W$ such that wRw' and $\mathcal{M}, w' \models \phi$
- $\mathcal{M}, w \models \Box\phi$ iff, for every $w' \in W$, wRw' implies $\mathcal{M}, w' \models \phi$.

In Kripke semantics, modal iteration translates into a ramification:

$$\mathcal{M}, w \vDash \Box\Box p \text{ iff } \forall v \text{ s.t. } wRv, \forall u \text{ s.t. } vRu, \mathcal{M}, u \vDash p.$$

The unfolding of the graph of R discharges the interpretation of modal iteration: the higher the modal degree of the formula, the longer the branches of the graph to consider.

Kripke semantics = optimal way to allow for modal iteration while maintaining the totalization of possible worlds.

Objection to Kripke

There is no reason to set once and for all the collection of possible worlds as a closed totality. In fact, there is every reason not to. Still, this is what Kripke semantics does.

Defense of Kripke 1: To different Kripke frames correspond, in general, different sets of possible worlds.

Answer: For each given interpretation, the set of all possible worlds is fixed. Validity is a metalinguistic predicate, not second-order necessity.

Defense of Kripke 2: The accessibility relation R induces the progression $w_0, E_{w_0} = \{w : w_0 R w\}, (E_w)_{w \in E_{w_0}}, \ldots$ (n-accessibility neighborhoods).

Answer:

- Any range of worlds is contained in advance in W.
- Nothing relates two different subsets E_w and $E_{w'}$ as making up some higher-order system of possibility.
- A possible world is not intrinsically relative to some system of possibility: It can lie at the end of accessibility lines of different lengths, so that levels of possibility cannot really be distinguished.

Provisional conclusion: Kripke admits iterated modalities and yet remains too close to Leibniz's conception of possible worlds.

Modal iteration should commit one to far more than is acknowledged in Kripke semantics.

The intuition of ranges of possibles ranges . . . of possible ranges of possible worlds is at the core of modal reasoning.

There is no easily tractable set-theoretic option to get what is wanted (namely, an open-ended collection of nested systems of possibility).

Moreover, the description of what is wanted faces the following
Problem: How could a mere set of possible worlds could make up a higher-order possible world (i.e. something structured and unified as a world)? How to make up a world with a multiplicity of worlds?

Solution: We should turn things upside down.
Let's rather think of each possible world as being virtually a set of possible worlds of higher-level (instead of thinking of a world of higher-level as being a set of worlds of lower-level).

Guiding picture:

- Higher-level possible worlds are specifications of lower-level worlds.
- Each lower-level world is the set of all its possible specifications at the next level. (Think of all possible Taylor approximations of functions having some given value at 0.)

Analysis, not synthesis: A world becomes a multiplicity of worlds insofar as the analysis of a certain structure at that world uncovers such a multiplicity.

A higher-order possible world is a basic (level 0) world, viewed as the index of some space containing higher-level possible worlds. (Distinction level/order.)

Wanted 2: A semantic framework

- comprising an open-ended collection of possible worlds of increasing levels
- where, at each modal iteration, a coherent set of possible worlds (system of possibility) is considered relatively to some already given possible world
- so that the interpretation of a formula of modal degree k involves nested worlds of respective levels $0, 1, \dots, k$ and of respective orders $k, k-1, \dots, 0$.

Candidate: a geometric manifold.

Manifold = generalization (to higher dimensions) of the notion of surface. Basic example: a curve, or a surface (e.g., the sphere). Notions of tangent vector (derivative of a curve), tangent space: see BLACKBOARD.

Many cues for differential geometry:

- To each point x of M is attached the tangent space to M at x, written T_xM. T_xM is composed of vectors.
- TM, the disjoint union of all the tangent spaces, is itself a manifold.
- Iterability: Hence $T^2M := TTM, T^3M, \ldots$ can be introduced.
- Change of scale: TM is twice as dimensional as M.
- The order of a world becomes comparable to the order of a Taylor expansion.

Basic scheme:

- 0-level possible worlds = points x of a given manifold M
- 1-level worlds relative to $x \in M$ = members of $T_x M$ (then x is considered as being of order 1)
- TM = set of all 1-level worlds in general
- accessible worlds = path-connected points (for a certain selection of paths)
- interpretation of a propositional variable p = set of curves on M
- p is true at x iff one of these curves passes through x
- interpretation of a formula ϕ of modal degree k = set of curves on $T^k M$
- ϕ is true at x iff one of these curves passes, not through x (impossible), but through some point of a set of points $\pi^k(x) \subseteq T^k M$ connected to x.

TASKS:

1. Interpreting of $\neg\phi$, $(\phi \wedge \psi)$ and $(\phi \vee \psi)$.
2. Defining modal lift: curves on M interpreting ϕ lifted into curves on TM interpreting $\Box\phi$.
3. Defining nonmodal lift. Example of $(\Box p \wedge q)$: the curves interpreting q have to be lifted onto TM to become homogeneous to the curves interpreting $\Box p$. Yet $\Box q$ does not occur in the formula.
4. Associating, to each point $x \in M$, a sequence $\pi^0(x) = \{x\}, \pi^1(x) \subseteq TM, \ldots, \pi^k(x) \subseteq T^k M, \ldots$ of counterparts of x
5. Checking validities.

Need of some further structure on M!

Technical definitions

Riemannian manifold = manifold M endowed with a *metrics*, that makes it possible to measure the distance between any two points of M, and thus to define which are the "straight lines" on M (geodesics).

Any metrics on M gives rise to a natural connection ∇, which allows one to connect vectors from some tangent space to vectors from some other tangent space.

Intuitively, ∇ is the operator that assigns, to any couple X, Y of vector (fields), the infinitesimal deviation of Y w.r.t. the direction indicated by X.

A vector field X defined along a curve γ is said to be parallel if $\nabla_X X = 0$ at any point of γ.

(A geodesic of $\langle M, g \rangle$ is nothing but a curve γ on M whose derivative γ' is parallel: It is a curve whose speed vector never deviates from itself.)

There is a natural projection $p : TM \to M$ which sends any tangent vector $v \in T_x M$ to the point x of M at which it is tangent.

Fact: For any geodesic $\gamma : \mathbb{R} \to M$ on M and any tangent vector v at $\gamma(0)$, there exists (locally) a unique curve $\delta : \mathbb{R} \to TM$ on TM such that:

- δ lies above γ (i.e. $p \circ \delta = \gamma$)
- δ passes through v (i.e. $\delta(0) = v$)
- δ (viewed as a vector field along γ) is parallel.

The curve δ is called the horizontal lift of γ through v at $t = 0$, and written $\widetilde{\gamma}^v$.

Fact: Any metrics g on M gives rise to a natural metrics g_T on TM.

Iteration: Starting from a Riemannian manifold $\langle M, g\rangle$,

- $M_0 = M$ and $g_0 = g$
- $M_{n+1} = T(M_n)$ and $g_{n+1} = (g_n)_T$
- $p_{n+1} : \langle M_{n+1}, g_{n+1}\rangle \to \langle M_n, g_n\rangle$ is the natural projection.

Last fact: As mentioned, the connection ∇ allows one to define

the parallel transport along a curve γ of any vector v in $T_{\gamma(t)}M$ into a vector $J^{\gamma}_{t,t'}(v)$ in $T_{\gamma(t')}M$.

The parallel transport connects possible worlds of the same level but lying above different points (= belonging to different systems of possibility).

End of technical definitions

TASK 4: Defining a sequence of counterparts of x above each $x \in M$.

To that end, $\langle M, g \rangle$ is endowed with a family $\mathcal{A} = \{\gamma_i : i \in I\}$ of accessibility curves on M.

Set of accessible worlds from x:

$$A^1(x) := \bigcup_{x \in \overline{\gamma_i}} \overline{\gamma_i}$$

($\overline{\gamma} := \{\gamma(t) : t \in \mathbb{R}\}$ = set of all the points of the curve γ. Likewise, for any set of curves Γ, $\overline{\Gamma} := \bigcup_{\gamma \in \Gamma} \overline{\gamma}$.)

Idea: The counterparts of x in TM come from the infinitesimal data at x sending x to a world accessible from x.

There is a way of coding the points of M accessible from x with elements of TM. For $v \in T_x M$, let c_v be the geodesic of M starting at x with speed v.

$P^1(x) := \{v \in B_x : c_v(1) \in A^1(x)\}$ = set of proxies of x in $T_x M$.

$\pi^1(x) := \{c_v'(t) : v \in P^1(x), t \in \mathbb{R}\}$ = set of all 1-counterparts of x.

Remark 1:

1-counterparts of $x \neq$ possible worlds of level 1 relative to x
The latter are all the elements of TM contained in the modal lift (cf. below) of a curve passing through x.

Remarks 2:

- The set $\pi^1(x)$ consists of curves on TM.
  The map $(v,t) \mapsto c'_v(t)$ is called the "geodesic flow" of $\langle M, g\rangle$.
- The construction of π^1 can be iterated:

  $$\pi^2 : TM \to \{\text{curves of } TTM\}$$

  Indeed, it suffices to replace $\langle M, g\rangle$ with $\langle TM, g_T\rangle$, and to replace the accessibility curves γ_i on M with the 2-accessibility curves on TM $J_w^{\gamma_i} : t \mapsto J_{t_w,t}^{\gamma_i}(w)$, for all $w \in \pi^1(x)$ (here t_w is the parameter such that $w \in T_{\gamma_i(t_w)}M$).
- The connection attached to the Riemannian manifold M is what supports the lift of accessibility curves on M into accessibility curves on TM. This expresses the geometric meaning taken on by accessibility here.
- The elements of $\pi^n(x)$ will be called the *n*-counterparts of x.

TASKS 2 & 3: Defining the modal and nonmodal lifts of any curve on M.

To that end, $\langle M, g \rangle$ is endowed, for each $n \geq 0$, with a selection $\mathcal{C}_n$ of admissibles curves on $T^n M$.

The members of $\mathcal{C}_n$ are all the curves from which the interpretation of any formula of modal degree n is bound down to being drawn.

One only asks:

- for any $v \in T^n M$ there exists at least one $\gamma \in \mathcal{C}_n$ passing through v
- $\delta \in \mathcal{C}_{n+1}$ implies $p_{n+1}(\delta) \in \mathcal{C}_n$
- for any $\gamma \in \mathcal{C}_n$, there exists at least one $\delta \in \mathcal{C}_{n+1}$ lifting γ (i.e. $p_{n+1}(\delta) = \gamma$).

Then, any curve γ on T^nM gives rise to two sets of curves on $T^{n+1}M$:

$$\lambda(\Gamma) = \{\widetilde{\gamma}^{k\gamma'(t)} \in \mathcal{C}_{n+1} : \gamma \in \Gamma, k \in \mathbb{R}^*, t \in \mathbb{R}\}$$

$$L_n^{n+1}(\Gamma) = \{\gamma' \in \mathcal{C}_{n+1} : \gamma \in \Gamma\}$$

Since the gap in modal degree between two principal subformulas of a single formula can be > 1, iterated nonmodal lift has to be defined:

$$L_n^n = \mathrm{id}, L_n^{n+m} = L_{n+m-1}^{n+m} \circ \ldots \circ L_n^{n+1}.$$

A metric modal frame $\underline{F}$ is a quadruple $\langle M, g, \mathcal{A}, (\mathcal{C}_n)_{n \geq 0} \rangle$.

A valuation V on $\underline{F}$ is the assignment, to any propositional variable p, of a subset $V(p)$ of $\mathcal{C}_0$ (closed under connected portions of its members).

A metric modal model is a metric modal frame endowed with a valuation.

TASK 1: Inductive definition of $V(\phi)$

- $V(\neg\phi) = \mathcal{C}_n \setminus V(\phi)$ (where $\deg(\phi) = n$)
- $V(\phi \wedge \psi) = V(\phi) \cap L^n_m(V(\psi))$ (where $\deg(\phi) = n$ and $\deg(\psi) = m < n$)
- $V(\phi \vee \psi) = V(\phi) \cup L^n_m(V(\psi))$ (under the same hypothesis as above)
- $V(\Diamond\phi) = \lambda(V(\phi))$

For any formula ϕ of modal degree n and for any $x \in M$:

$$\langle \underline{F}, V\rangle, x \vDash \phi \;\text{ iff }\; \pi^n(x) \cap \overline{V(\phi)} \neq \emptyset.$$

In other words, a modal formula ϕ is true at x if one of the possible worlds relative to x that are prompted by the construction of $V(\phi)$ coincides with some counterpart of x at the level defined by the modal degree of ϕ.

Simple case

Let $\mathbb{E}^2$ be the Euclidean plane endowed with a referential, with all the horizontal lines as the accessibility curves.
Let $O = (0,0)$ be the point taken as the origin of $\mathbb{E}^2$.
The unique accessibility curve passing through O, by hypothesis, is the x-axis, so $A^1(O) = \{(a,0) : a \in \mathbb{R}\}$. Thus
$P^1(O) = \{((0,0), \begin{pmatrix} \alpha \\ 0 \end{pmatrix} : \alpha \in \mathbb{R}\}$, hence
$\pi^1(O) = \{((a,0), \begin{pmatrix} \alpha \\ 0 \end{pmatrix}) : a \in \mathbb{R}, \alpha \in \mathbb{R}\}$.

One finally gets:

$\mathbb{E}^2, O \models \Diamond p$ iff there exists a curve γ_p in $V(p)$ that crosses the x-axis at least once (even if it is not at O).

This last clause is reminiscent of Kripke semantics. Actually, each Kripke frame $\mathcal{K} = \langle W, R \rangle$ corresponds to the metric frame $\underline{F}^{\mathcal{K}} = \langle M^{\mathcal{K}}, g^{\mathcal{K}}, \mathcal{A}^{\mathcal{K}}, (\mathcal{C}_n^{\mathcal{K}})_{n \geq 0} \rangle$ defined by:

- $M^{\mathcal{K}} = W$ as a discrete topological space
- $g^{\mathcal{K}}$ is the null metrics
- $\mathcal{A}^{\mathcal{K}}$ is the set of all maximal paths of the graph of R
- $\mathcal{C}_n^{\mathcal{K}}$ is the set of all curves on $T^n W \cong W$.

Kripke semantics amounts to disregarding the space that underpins the web of accessibility relations between the possible worlds.

The modal semantics based on metric models thus generalizes Kripke semantics.

Modal geometry introduces some structure which lumps all the possible worlds together and gives them the texture of an underlying geometric universe,
by determining the accessibility relations and the evaluation to proceed in an essentially local way.

Remark: Two formulae ϕ and ψ, interpreted as sets of curves, may have coextensional valuations (pass through the same 0-worlds) and yet have non coextensional possibilizations $V(\phi)$ and $V(\psi)$
This is because each curve takes into account the way in which it passes through each world that it contains.

Three kinds of modal semantics

The number of matches between an evaluation world x and a valuation $V(\Diamond\phi)$ is supposed to be generally greater than the number of matches between x and $V(\phi)$. There are three way to achieve this:

- $V(\Diamond\phi)$ enlarges $V(\phi)$, while the evaluation world x is left unchanged.
  This is what happens in Tarski-McKinsey semantics, where $V(\Diamond\phi) =$ topological closure of $V(\phi)$.
- $V(\phi)$ is not modified, but the shift to $\Diamond\phi$ results in x being enlarged into a whole set of worlds.
  That is what happens in Kripke semantics.
- Both $V(\phi)$ and x are enlarged.
  That is what happens in the present framework: $V(\Diamond\phi)$ enlarges $V(\phi)$, and x is enlarged into $\pi^1(x)$ as well, while everything is transferred to the next level (in the progression $\ldots T^2M \to TM \to M$).

TASK 5: Validity and completeness results.

Axioms **T** and **K** are valid.

The axiom **4** is not valid in general.
Indeed, $\langle \underline{F}, V\rangle, x \vDash \Diamond p$ obtains if $\lambda(V(p))$ passes through at least one point in $\pi(x)$. On the contrary, $\langle \underline{F}, V\rangle, x \vDash \Diamond\Diamond p$ obtains simply if $\lambda(\lambda(V(p)))$ passes through at least one point in $\pi(w)$ for some element w of $\pi(x)$ that can perfectly be other than x.

However, the axiom **4** becomes valid if one restricts oneself to a class of very special metric modal frames.

Definition

A metric (modal) frame $\underline{F} = \langle M, g, \mathcal{A}, (\mathcal{C}_n)_{n\geq 0}\rangle$ is sparse if the following conditions hold:

(i) $\langle M, g\rangle$ is a simple Riemannian manifold (i.e. any two points of M are related by at most one geodesic)

(ii) $\mathcal{A}$ and $\mathcal{C}_0$ are composed of geodesics only

(iii) for any $n \geq 0$, $\mathcal{C}_{n+1}$ is the set of all the horizontal lifts of curves in $\mathcal{C}_n$.

Proposition

The axiom **4** *is valid in all sparse metric frames.*

Modus ponens obviously preserves validity in sparse metric frames.

So does the necessitation rule, but on the condition that sparse metric frames are replaced with a slightly stricter class of metric frames (see below).

The rule of uniform substitution does not preserve validity, but that is in order.
Indeed, when one is committed to some modal claim (such as $\Box\Box p \rightarrow \Box p$), one should not be committed to the analogous claim about formulae of higher modal degree than that of p.
Closure under substitution is motivated by the belief that if one is able to establish something about simple necessity, one is by the same token entitled to admit the same thing about necessary necessity.
But the claim that the meaning of necessity is uniform is not self-evident at all.

S4* = S4 with the rule of uniform substitution restricted to the case where the substituens is of modal degree not higher than that of the substituendum.

S4** = S4* without the necessitation rule.

By the previous proposition, the system S4** is valid in all sparse metric frames. There is more.

Proposition

*The system S4** is complete w.r.t. the class of all sparse metric frames.*

A completeness result can also be reached for S4*.

Definition

An Hadamard metric frame is a metric frame $\underline{F} = \langle M, g, \mathcal{A}, (\mathcal{C}_n)_{n\geq 0}\rangle$ such that:

(i) $\langle M, g\rangle$ is a simply connected, complete Riemannian manifold of nonpositive sectional curvature

(ii) $\mathcal{A}$ and $\mathcal{C}_0$ are composed of geodesics only

(iii) for any $n \geq 0$, $\mathcal{C}_{n+1}$ is the set of all the horizontal lifts of curves in $\mathcal{C}_n$.

Theorem

The system S4 is complete w.r.t. the class of all Hadamard metric frames.*

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Thus, the partial neutralization of modal iteration expressed by the axiom **4** corresponds to a really very special class of metric frames (more so than in Kripke semantics).

This result is in line with the principle of geometric modal logic, based on the project of giving modal iteration its full meaning.

S5* = S5 with the rule of uniform substitution restricted as in the case of S4*.

Definition

A sparse metric frame $\underline{F} = \langle M, g, \mathcal{A}, (\mathcal{C}_n)_{n \geq 0} \rangle$ is minimal if any two geodesics in $\mathcal{A}$ are either disjoint or identical (up to linear reparameterization).

Proposition

The axiom **5** *is valid in all minimal metric frames.*

Theorem

The system S5 is complete w.r.t. the class of all minimal metric frames.*

Conclusion

(I) Conceptual thesis:

- A more faithful account of modal iteration does not imply abandoning a possible worlds semantics, but it surely implies abandoning the Leibnizian heritage of an absolute, complete totality of all possible worlds.
  Despite the introduction of accessibility relations, Kripke semantics still endorses that heritage.
- The collection of all actually possible worlds should be itself counterfactualizable.
  Such a representation triggers the consideration of an open-ended collection of possible worlds organized along indefinitely many increasing levels, so that a genuine change of scale occurs at each step.
  A notion of higher-order possible world is needed.

(II) Technical implementation: The second part of my talk has been devoted to building up a semantic framework in keeping with that conceptual thesis.

- The resulting framework is technically complex (more difficult to handle than Kripke semantics). But that is not the point.
- It disposes of any complete totality of possible worlds.
- It generalizes in a certain way Kripke semantics. Kripke semantics amounts to taking a uniformly flat space to underpin the web of accessibility relations between possible worlds.
- It shows interesting interactions with geometry. Similarity between modal operators and differential operators.
- It allows one to establish completeness results about variants of S4 and S5. These results show the peculiarity of the axioms (like axioms **4** and **5**) which neutralize modal iteration.